

# KOSZUL PROPERTY AND BOGOMOLOV'S CONJECTURE

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## 1. INTRODUCTION

**1.1.** Let  $F$  be an arbitrary field and  $G_F = \text{Gal}(\bar{F}/F)$  be the Galois group of its (separable) algebraic closure  $\bar{F}$  over it. Two conjectures about the homological properties of the group  $G_F$  are widely known. First of them, the *Milnor–Kato conjecture*, claims that for any prime number  $l \neq \text{char } F$  the algebra of Galois cohomology with cyclotomic coefficients

$$H^*(G_F, \mu_l^{\otimes *}) = \bigoplus_{n=0}^{\infty} H^n(G_F, \mu_l^{\otimes n})$$

is generated by  $H^1$  and defined by quadratic relations. Here, as usually, we denote by  $\mu_l$  the group of  $l$ -roots of unity in  $\bar{F}$ .

More precisely, there is a canonical homomorphism of graded algebras called the *Galois symbol*, or the *norm residue homomorphism*

$$K_*^M(F) \otimes \mathbb{Z}/l \longrightarrow H^*(G_F, \mu_l^{\otimes *})$$

from the Milnor K-theory ring  $K_*^M(F)$  of the field  $F$  modulo  $l$  to the cyclotomic Galois cohomology. One defines this homomorphism in degree 1 as the classical Kummer isomorphism  $F^*/F^{*l} \simeq H^1(G_F, \mu_l)$ , then shows that it can be extended to the higher degrees by multiplicativity [3]. The conjecture says that for any field  $F$  and any  $l \neq \text{char } F$  this map is an isomorphism [12, 10].

A. Merkurjev and A. Suslin proved that it is an isomorphism in degree  $n = 2$ ; later the same authors and, independently, M. Rost found a proof for  $l = 2$  and  $n = 3$ . Recently V. Voevodsky [21] obtained a general proof for  $l = 2$ .

**1.2.** The second conjecture is due to F. Bogomolov [7]. It claims that, whenever the field  $F$  contains an algebraically closed subfield, the commutator subgroup of the Sylow pro- $l$ -subgroup of  $G_F$  is a free pro- $l$ -group. Furthermore, the commutator subgroup of the maximal quotient pro- $l$ -group of  $G_F$  should be a free pro- $l$ -group as well. The latter statement is stronger, because the former one can be deduced from it by passing from  $F$  to the field corresponding to the Sylow subgroup of  $G_F$ .

The condition about algebraically closed subfield cannot be dropped, though it appears that the only essential part of it is that the field  $F$  should contain all the roots of unity of degrees equal to powers of  $l$ . Indeed, in the appendix we construct simple counterexamples showing that both the above formulations of the conjecture fail for fields without these roots of unity. Nevertheless, the author thinks that there is a way to extend Bogomolov's conjecture to all fields.

**Conjecture 1.** *Let  $F$  be an arbitrary field and  $l$  be a prime number. Let  $K = F[\sqrt[l^\infty]{F}]$  be the field obtained by adjoining to  $F$  all roots of degrees  $l^N$ ,  $N > 0$ , of all the elements of  $F$ . Then the Sylow pro- $l$ -subgroup of the absolute Galois group  $G_K$  of the field  $K$  is a free pro- $l$ -group.*

It is well-known that the statement of Conjecture 1 is true for any field of characteristic  $l$  [18]; in the sequel we will presume that we are not in this case. Besides, the conjecture is true for all number fields and their functional analogues (where it is even sufficient to adjoin the roots of unity). Conjecture 1 holds for a complete discrete valuation field whenever it holds for the residue field. This is the most important evidence supporting Conjecture 1 that is currently known to the author.

**1.3.** Conjecture 1 is very strong and likely hard to approach. The following weaker version is closer to original Bogomolov's conjecture.

**Conjecture 2.** *Let  $F$  be a field containing a primitive root of unity of degree  $l$  if  $l$  is odd, and a field containing a square root of  $-1$  if  $l = 2$ . Then the maximal quotient pro- $l$ -group of the absolute Galois group  $G_K$  of the field  $K = F[\sqrt[l^\infty]{F}]$  is a free pro- $l$ -group.*

Conjecture 2 follows from Conjecture 1, because the maximal quotient pro- $l$ -group  $G^{(l)}$  of a pro-finite group  $G$  is free whenever the Sylow pro- $l$ -subgroup  $S_l G$  of  $G$  is free. Indeed, both maps in the sequence  $H^2(G^{(l)}, \mathbb{Z}/l) \rightarrow H^2(G, \mathbb{Z}/l) \rightarrow H^2(S_l G, \mathbb{Z}/l)$  are easily seen to be injective for any pro-finite group  $G$ .

**1.4.** The relation between the Milnor–Kato and freeness conjectures is the main topic of Bogomolov's paper [7]. It seems that no one of the two conjectures implies directly the other one; rather, they are somewhat complementary.

**1.5.** The approach to the Milnor–Kato and freeness conjectures which we develop here was started in the paper [14] by Alexander Vishik and me. There it was proven that the statement of the Milnor–Kato conjecture for a field  $F$  whose Galois group  $G_F$  is a pro- $l$ -group would follow from its low-degree part if we knew the Milnor K-theory ring  $K_*^M(F) \otimes \mathbb{Z}/l$  to be a *Koszul algebra*. The following theorem, which we prove in section 5 (as the first statement of Corollary 2) slightly generalizes the main result of [14].

**Theorem 1.** *Let  $l$  be a prime number and  $F$  be a field containing a primitive  $l$ -root of unity. Suppose that*

- (1) *the map  $K_n^M(F) \otimes \mathbb{Z}/l \rightarrow H^n(G_F, \mu_l^{\otimes n})$  is an isomorphism for  $n = 2$  and a monomorphism for  $n = 3$ ;*
- (2) *the algebra  $K_*^M(F) \otimes \mathbb{Z}/l$  is Koszul.*

*Let  $G_F^{(l)}$  be the maximal quotient pro- $l$ -group of the group  $G_F$ . Then the natural homomorphism  $K_*^M(F) \otimes \mathbb{Z}/l \rightarrow H^*(G_F^{(l)}, \mu_l^{\otimes *})$  is an isomorphism.*

Of course, the first part of the condition (1) is the theorem of Merkurjev and Suslin; the second part of (1) is long known to be true, at least, for  $l = 2$ . It is not difficult

to show, using the transfer operations on the Milnor K-theory, that it suffices to verify the Milnor–Kato conjecture in the pro- $l$ -group case [7, 20]. Thus our Koszulity hypothesis essentially implies the Milnor–Kato.

**1.6.** The main goal of this paper is to show that a certain much stronger hypothesis about the Milnor ring would imply Conjecture 2 (Bogomolov’s conjecture) as well.

Under the conditions of Conjecture 2, the Milnor K-theory algebra modulo  $l$  is a quotient algebra of the exterior algebra of the vector space  $F^*/F^{*l}$  over the prime field  $\mathbb{Z}/l$ , so we can consider the natural homomorphism

$$\Lambda_l^*(F) = \bigwedge_{\mathbb{Z}/l}^*(F^*/F^{*l}) \longrightarrow K_*^M(F) \otimes \mathbb{Z}/l.$$

Notice that the exterior algebra  $\Lambda_l^*(F)$  is isomorphic to the cohomology algebra of the Galois group  $\text{Gal}(F[\sqrt[l]{F}]/F)$  with coefficients  $\mathbb{Z}/l$  (see section 9).

The stronger Koszulity hypothesis states that the kernel  $J_l^*(F)$  of this map should be a *Koszul module* over the exterior algebra (see section 3 for the definition). Our main result can be formulated as follows.

**Theorem 2.** *Let  $l$  be a prime number and  $F$  be a field containing the  $l$ -roots of unity if  $l$  is odd, and containing the 4-roots of unity if  $l = 2$ . Suppose that*

- (1) *the condition (1) of Theorem 1 holds;*
- (2) *the ideal  $J_l^*(F)$  generated by the Steinberg relations in the exterior algebra  $\Lambda_l^*(F)$  is a Koszul module (in the grading shifted by 2) over the algebra  $\Lambda_l^*(F)$ .*

*Then the maximal quotient pro- $l$ -group of the absolute Galois group  $G_K$  of the field  $K = F[\sqrt[l]{F}]$  is a free pro- $l$ -group.*

It is shown in section 6 that the condition (2) of Theorem 1 follows from the condition (2) of Theorem 2. Therefore, in the assumptions of Theorem 2 both the Milnor–Kato and Bogomolov’s conjectures are true.

**1.7.** Two proofs of Theorem 2 are given in this paper. The first one is rather computational; it is based on the Serre–Hochschild spectral sequence for pro- $l$ -groups and a result extending the techniques of [14] to cohomology of conilpotent comodules. The latter is proven in section 4. The second argument uses the results of [14] in order to pass to the associated graded coalgebras and then applies directly a general principle claiming that Koszulity-type conditions on morphisms of Koszul algebras correspond to freeness-type conditions under duality. This is explained in section 7.

**1.8.** Finally, let us point to a simple application of our results to a different area of motivic theory—that of mixed Tate motives with rational coefficients over a field of finite characteristic. For such a field  $F$ , the standard conjectures [4] imply that the Milnor K-theory algebra with rational coefficients  $K_*^M(F) \otimes \mathbb{Q}$  should coincide with the algebraic K-theory  $K_*(F) \otimes \mathbb{Q}$  and be a Koszul algebra. Furthermore, in this case the *motivic pro-Lie algebra*  $\mathfrak{L}_*(F) = \bigoplus_{i=1}^{\infty} \mathfrak{L}_i(F)$  describing the abelian tensor category of mixed Tate motives with rational coefficients over a field  $F$  is isomorphic to the quadratic pro-Lie algebra dual to  $K_*^M(F) \otimes \mathbb{Q}$ . (The duality here connects the quadratic pro-Lie algebras with the skew-commutative quadratic algebras.)

A conjecture of A. Goncharov [9] claims that for any field  $F$  the subalgebra  $\mathfrak{L}_{\geq 2}(F) = \bigoplus_{i=2}^{\infty} \mathfrak{L}_i(F)$  of the motivic pro-Lie algebra  $\mathfrak{L}_*(F)$  is a free graded pro-Lie algebra. It follows from our result that this conjecture together with the Koszulity conjecture above is equivalent to the following hypothesis about the Milnor K-theory algebra. Let  $J_{\mathbb{Q}}^*(F)$  be the kernel of the homomorphism from the exterior algebra  $\bigwedge_{\mathbb{Q}}^*(F^* \otimes \mathbb{Q})$  to  $K_*^M(F) \otimes \mathbb{Q}$ ; then the ideal  $J_{\mathbb{Q}}^*(F)$  should be a Koszul module over the exterior algebra. Note that in this setting, unlike in the Bogomolov case, the analogue of the statement converse to Theorem 2 also holds: the algebra  $K_*^M(F) \otimes \mathbb{Q}$  is Koszul and the pro-Lie algebra  $\mathfrak{L}_{\geq 2}(F)$  is free if and only if the ideal  $J_{\mathbb{Q}}^*(F)$  is a Koszul module. The basic reason is that  $\mathfrak{L}_*(F)$  is a graded pro-Lie algebra, while in the Bogomolov case we deal with an ungraded Galois group. In this sense, the hypothesis of Koszulity of the Steinberg ideal  $J_l^*(F)$  can be thought of as a stronger graded version of Conjecture 2. This is explained in section 8.

**1.9.** Coalgebras, comodules, and their cohomology are considered in section 2. Definitions and basic properties of Koszul (co)algebras and (co)modules are presented in section 3. The cohomology of conilpotent comodules are studied in section 4, the cohomology of nonconilpotent coalgebras in section 5. We discuss Koszul properties related to morphisms of graded algebras in section 6. The duality between Koszulity and freeness conditions is explained in section 7. Koszulity and freeness for Lie coalgebras are considered in section 8. We prove the main theorem in section 9. Evidence related to Conjecture 1 is presented in the appendix.

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## 2. HOMOLOGY OF ALGEBRAS AND COHOMOLOGY OF COALGEBRAS

**2.1. The cohomology of comodules.** Recall that a *coalgebra*  $C$  over a field  $\mathbb{k}$  is a vector space equipped with a comultiplication map  $\Delta: C \rightarrow C \otimes C$  and a counit map  $\varepsilon: C \rightarrow \mathbb{k}$  satisfying the conventional coassociativity and counit axioms. A *coaugmented coalgebra* is a coalgebra  $C$  endowed with a coalgebra homomorphism  $\gamma: \mathbb{k} \rightarrow C$ . The cohomology algebra  $H^*(C)$  of a coaugmented coalgebra  $C$  is the opposite multiplication algebra to the Ext-algebra  $\text{Ext}_C^*(\mathbb{k}, \mathbb{k})$ , where  $\mathbb{k}$  is endowed with the left  $C$ -comodule structure by means of  $\gamma$ .

A left *comodule*  $P$  over a coalgebra  $C$  is a vector space over  $\mathbb{k}$  together with a coaction map  $\Delta': P \rightarrow C \otimes P$  satisfying the usual coaction axiom. The cohomology

module of a comodule  $P$  over a coaugmented coalgebra  $C$  is the left  $H^*(C)$ -module  $H^*(C, P) = \text{Ext}_C^*(\mathbb{k}, P)$ . This cohomology can be calculated in terms of the following explicit cobar-resolution

$$P \longrightarrow C \otimes P \longrightarrow C \otimes C^+ \otimes P \longrightarrow C \otimes C^+ \otimes C^+ \otimes P \longrightarrow \dots,$$

where  $C^+ = \text{coker } \gamma$  and the cobar-differential is given by the well-known formula  $d(c_0 \otimes \dots \otimes c_n \otimes p) = \sum_{i=0}^n (-1)^{i-1} c_0 \otimes \dots \otimes \Delta(c_i) \otimes \dots \otimes c_n \otimes p + (-1)^n c_0 \otimes \dots \otimes c_n \otimes \Delta'(p)$ . It is easy to check that this is an injective comodule resolution of  $P$ . Applying the functor  $\text{Hom}_C(\mathbb{k}, -)$  to it, we get

$$H^*(C, P) = H^*(P \longrightarrow C^+ \otimes P \longrightarrow C^+ \otimes C^+ \otimes P \longrightarrow \dots)$$

and in particular

$$H^*(C) = H^*(\mathbb{k} \longrightarrow C^+ \longrightarrow C^+ \otimes C^+ \longrightarrow \dots).$$

The multiplication on  $H^*(C)$  and the action of  $H^*(C)$  on  $H^*(C, P)$  are induced by the obvious multiplication and action

$$\begin{aligned} (c_1 \otimes \dots \otimes c_i) \cdot (c_{i+1} \otimes \dots \otimes c_{i+j}) &= c_1 \otimes \dots \otimes c_i \otimes c_{i+1} \otimes \dots \otimes c_{i+j} \\ (c_1 \otimes \dots \otimes c_i) \cdot (c_{i+1} \otimes \dots \otimes c_{i+j} \otimes p) &= c_1 \otimes \dots \otimes c_i \otimes c_{i+1} \otimes \dots \otimes c_{i+j} \otimes p \end{aligned}$$

on the level of cobar-complexes.

**2.2. The homology of modules.** The homology coalgebra  $H_*(A)$  of an augmented algebra  $A$  is defined as  $H_*(A) = \text{Tor}_*^A(\mathbb{k}, \mathbb{k})$ , where the left and the right  $A$ -module structures on  $\mathbb{k}$  are defined by means of the augmentation morphism  $\alpha: A \longrightarrow \mathbb{k}$ . Let  $M$  be a left module over an augmented algebra  $A$ . The homology comodule of  $M$  is the left  $H_*(A)$ -comodule  $H_*(A, M) = \text{Tor}_*^A(\mathbb{k}, M)$ . This homology can be calculated by means of the bar-resolution

$$M \longleftarrow A \otimes M \longleftarrow A \otimes A_+ \otimes M \longleftarrow A \otimes A_+ \otimes A_+ \otimes M \longleftarrow \dots$$

with  $A_+ = \ker \alpha$  and the standard bar-differential. Therefore we have

$$H_*(A, M) = H_*(M \longleftarrow A_+ \otimes M \longleftarrow A_+ \otimes A_+ \otimes M \longleftarrow \dots)$$

and in particular

$$H_*(A) = H_*(\mathbb{k} \longleftarrow A_+ \longleftarrow A_+ \otimes A_+ \longleftarrow \dots).$$

The comultiplication on  $H_*(A)$  and the coaction of  $H_*(A)$  on  $H_*(A, M)$  are induced by the obvious comultiplication and coaction

$$\begin{aligned} \Delta(a_1 \otimes \dots \otimes a_n) &= \sum_{i=0}^n (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n) \\ \Delta'(a_1 \otimes \dots \otimes a_n \otimes m) &= \sum_{i=0}^n (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_n \otimes m) \end{aligned}$$

on the level of bar-complexes.

**2.3. Graded modules and comodules.** By a *graded coalgebra* we mean a non-negatively graded vector space  $C = \bigoplus_{n=0}^{\infty} C_n$  equipped with a coalgebra structure which respects the grading, i. e.,  $\Delta(C_n) \subset \sum_{i+j=n} C_i \otimes C_j$  and  $\varepsilon(C_{>0}) = 0$ . Moreover, we will always assume the map  $\varepsilon : C_0 \rightarrow \mathbb{k}$  to be an isomorphism. Then a graded coalgebra structure induces a coaugmented coalgebra structure in the obvious way. A *graded comodule* over  $C$  is a graded vector space  $P = \bigoplus_{i=0}^{\infty} P_i$  equipped with a  $C$ -comodule structure which respects the grading, i. e.,  $\Delta'(P_n) \subset \sum_{i+j=n} C_i \otimes P_j$ . We always consider the nonnegatively graded comodules only, that is,  $P_i = 0$  for  $i < 0$ .

We leave it to the reader to formulate the dual definitions of graded algebras and modules; as above, we assume that  $A_i = 0$  for  $i < 0$ , that  $A_0$  is a one-dimensional vector space generated by the unit, and that  $M_i = 0$  for  $i < 0$ .

One can immediately see from the bar-complexes above that the homology and cohomology of graded objects are endowed with a natural second grading: for a graded algebra  $A$  and a graded  $A$ -module  $M$  we have

$$H_*(A) = \bigoplus_{i \leq j} H_{i,j}(A) \quad \text{and} \quad H_*(A, M) = \bigoplus_{i \leq j} H_{i,j}(A, M);$$

and analogously, for a graded coalgebra  $C$  and a graded  $C$ -comodule  $P$

$$H^*(C) = \bigoplus_{i \leq j} H^{i,j}(C) \quad \text{and} \quad H^*(C, P) = \bigoplus_{i \leq j} H^{i,j}(C, P).$$

**2.4. Opposite multiplications and comultiplications.** For algebras  $A$  and  $A^{\text{op}}$  with opposite multiplication there is a canonical isomorphism of graded vector spaces  $H_*(A) \simeq H_*(A^{\text{op}})$  given by the tautological isomorphism  $\text{Tor}_*^A(\mathbb{k}, \mathbb{k}) \simeq \text{Tor}_*^{A^{\text{op}}}(\mathbb{k}, \mathbb{k})$ . It is easy to see from the bar-complexes above that this isomorphism makes  $H_*(A)$  and  $H_*(A^{\text{op}})$  coalgebras with opposite comultiplication.

On the other hand, for coalgebras  $C$  and  $C^{\text{op}}$  with opposite comultiplication there is no self-obvious isomorphism between  $\text{Ext}_C^*(\mathbb{k}, \mathbb{k})$  and  $\text{Ext}_{C^{\text{op}}}^*(\mathbb{k}, \mathbb{k})$ . Nevertheless, one can use the cobar-complexes to construct an isomorphism of graded vector spaces  $H^*(C) \simeq H^*(C^{\text{op}})$  which will make them algebras with opposite multiplication.

To define this isomorphism without mentioning explicit resolutions, one has to introduce the *cotensor product* and *cotorsion* functors, as explained in the next subsection. This will complete the symmetry between the algebras and coalgebras.

**2.5. The cotorsion functors and cohomology of comodules.** Let  $P$  and  $Q$  be a right and a left comodule over a coalgebra  $C$ . The *cotensor product* vector space  $P \odot_C Q$  is defined as the kernel of the map  $\Delta' \otimes \text{id} - \text{id} \otimes \Delta' : P \otimes Q \rightarrow P \otimes C \otimes Q$ . It is not difficult to check that this operation satisfies the associativity and unit identities analogous to those for the tensor product of modules over algebras:  $(P \odot_C Q) \odot_D R = P \odot_C (Q \odot_D R)$  and  $C \odot_C P = P$ .

The cotensor product is a left-exact functor on both arguments; according to the above identities, the functors of cotensor product with cofree comodules are exact. As usual, it follows that the right derived functors of the functor of cotensor product along the first and the second arguments coincide; we will call them the *cotorsion*

*functors* and denote by  $\text{Cot}_C^i(P, Q)$ . Using the cobar-resolution from subsection 1.1, one can compute those derived functors explicitly:

$$\text{Cot}_C^*(P, Q) = H^*(P \otimes Q \longrightarrow P \otimes C \otimes Q \longrightarrow P \otimes C \otimes C \otimes Q \longrightarrow \cdots),$$

where  $C$  can be replaced with  $C_+ = \ker \varepsilon$ .

For any left  $C$ -comodule  $P$  and finite-dimensional left  $C$ -comodule  $Q$  there are natural isomorphisms  $\text{Ext}_C^i(Q, P) \simeq \text{Cot}_C^i(Q^*, P)$ , just as for any left  $A$ -module  $M$  and right  $A$ -module  $N$  one has  $\text{Ext}_A^i(M, N^*) \simeq \text{Tor}_i^A(N, M)^*$ . Therefore, we may alternatively define the cohomology of a coaugmented coalgebra as  $H^*(C) = \text{Cot}_C^*(\mathbb{k}, \mathbb{k})$  and the cohomology of a comodule as  $H^*(C, P) = \text{Cot}_C^*(\mathbb{k}, P)$ .

**Remark:** All the definitions of this section can be given for the (co)algebra and (co)module structures on (graded) objects of an arbitrary (not necessarily semisimple; associative, but not necessarily commutative) abelian tensor category (with an exact functor of tensor product). The same holds for all the definitions and results of section 3, 6 and 7; all of them are duality-symmetric, in particular. For the results of section 8 mentioning commutative and Lie structures, the tensor category has to be commutative, of course. All of this does not apply to the results of sections 4 and 5, however, which depend essentially on the standard properties of the functor of direct limit and therefore are not duality-symmetric at all. But we prefer to deal with vector spaces in this paper.

### 3. KOSZUL PROPERTY FOR MODULES AND COMODULES

Different accounts of the basic properties of *quadratic* and *Koszul algebras* can be found in the papers [16, 2, 5, 6]. The notion of a *Koszul module* and related results first appeared in [5]. In the infinite-dimensional case one has to define the *quadratic duality* as a correspondence between algebras and coalgebras [14]. The purpose of this section is to present the analogous results for modules and comodules.

**3.1. Quadratic modules and comodules.** A graded algebra  $A$  is called *one-generated* if it is multiplicatively generated by  $A_1$ . A graded algebra  $A$  is called *quadratic* if it is isomorphic to the quotient algebra  $\{V, R\} = \mathbb{T}(V)/(R)$  of the tensor algebra  $\mathbb{T}(V) = \bigoplus_n V^{\otimes n}$  of the vector space  $V = A_1$  by the ideal generated by a subspace  $R \subset V^{\otimes 2}$ . For any graded algebra  $A$  there exists a unique quadratic algebra  $\text{q}A$  together with a morphism of graded algebras  $r_A: \text{q}A \longrightarrow A$  which is an isomorphism on  $A_1$  and a monomorphism on  $A_2$ . To construct the algebra  $\text{q}A$ , one can take  $A_1$  for  $V$  and the kernel of the multiplication map  $A_1 \otimes A_1 \longrightarrow A_2$  for  $R$ .

A graded module  $M$  over a one-generated algebra  $A$  is called *one-generated* if it is generated by  $M_0$ , that is the map  $A \otimes M_0 \longrightarrow M$  is surjective. A graded module  $M$  over a quadratic algebra  $A = \{V, R\}$  is called *quadratic* if it is isomorphic to the quotient module  $\{U, S\}_A = A \otimes U / AS$  of the free  $A$ -module  $A \otimes U$  generated by the vector space  $U = M_0$  by the submodule generated by a vector subspace  $S \subset V \otimes U$ . For any graded module  $M$  over a graded algebra  $A$  there exists a unique

quadratic module  $q_A M$  over the quadratic algebra  $qA$  together with a morphism of graded  $qA$ -modules  $r_{A,M}: q_A M \rightarrow M$  which is an isomorphism on  $M_0$  and a monomorphism on  $M_1$ . To construct the module  $q_A M$ , one takes  $M_0$  for  $U$  and the kernel of the map  $A_1 \otimes M_0 \rightarrow M_1$  for  $S$ .

A graded coalgebra  $C$  is called *one-cogenerated* if the iterated comultiplication maps  $\Delta^{(n)}: C_n \rightarrow C_1^{\otimes n}$  are injective, or equivalently, all the maps  $\Delta: C_{i+j} \rightarrow C_i \otimes C_j$  for  $i, j \geq 0$  are injective. A graded coalgebra is called *quadratic* if it is isomorphic to the subcoalgebra of the tensor coalgebra  $\bigoplus_n V^{\otimes n}$  of the form

$$\langle V, R \rangle = \bigoplus_{n=0}^{\infty} \bigcap_{i=1}^{n-1} V^{i-1} \otimes R \otimes V^{n-i-1}$$

for the vector space  $V = C_1$  and a certain subspace  $R \subset V^{\otimes 2}$ . For any graded coalgebra  $C$  there exists a unique quadratic coalgebra  $qC$  together with a morphism of graded coalgebras  $r_C: C \rightarrow qC$  which is an isomorphism on  $C_1$  and an epimorphism on  $C_2$ . Indeed, it suffices to take  $C_1$  for  $V$  and the image of the comultiplication map  $C_2 \rightarrow C_1 \otimes C_1$  for  $R$ .

A graded comodule  $P$  over a one-cogenerated coalgebra  $C$  is called *one-cogenerated* (cogenerated by  $P_0$ ) if all the coaction maps  $\Delta': P_{i+j} \rightarrow C_i \otimes P_j$  for  $i, j \geq 0$  are injective, or equivalently, the maps  $\Delta': P_i \rightarrow C_i \otimes P_0$  are injective. A graded comodule  $P$  over a quadratic coalgebra  $C = \langle V, R \rangle$  is called *quadratic* if it is isomorphic to the subcomodule of the cofree  $C$ -comodule  $C \otimes U$  of the form

$$\langle U, S \rangle_C = \bigoplus_{n=0}^{\infty} C_n \otimes U \cap C_{n-1} \otimes S$$

for the vector space  $U = P_0$  and a certain subspace  $S \subset V \otimes U$ . For any graded comodule  $P$  over a graded coalgebra  $C$  there exists a unique quadratic comodule  $q_C P$  over the quadratic coalgebra  $qC$  together with a morphism of graded  $qC$ -comodules  $r_{C,P}: P \rightarrow q_C P$  which is an isomorphism on  $P_0$  and an epimorphism on  $P_1$ . Namely, one can take  $P_0$  for  $U$  and the image of the coaction map  $P_1 \rightarrow C_1 \otimes P_0$  for  $S$ .

**3.2. Quadratic duality.** The quadratic algebra  $A = \{V, R\}$  and the quadratic coalgebra  $C = \langle V, R \rangle$  are called *dual* to each other; we denote this as  $A = C^!$  and  $C = A^?$ . This rule defines an equivalence between the category of quadratic algebras and the category of quadratic coalgebras. Furthermore, the quadratic  $A$ -module  $M = \{U, S\}_A$  and the quadratic  $C$ -comodule  $P = \langle U, S \rangle_C$  are called *dual* to each other, too; the notation:  $M = P_C^!$  and  $P = M_A^?$ . This defines an equivalence between the category of quadratic  $A$ -modules and the category of quadratic  $C$ -comodules.

**Proposition 1.** *A graded coalgebra  $C$  is one-cogenerated if and only if  $H^{1,j}(C) = 0$  for all  $j > 1$ . A one-cogenerated coalgebra  $C$  is quadratic if and only if  $H^{2,j}(C) = 0$  for all  $j > 2$ . More precisely, for any graded coalgebra  $C$  the morphism  $r_C: C \rightarrow qC$  is an isomorphism in degree  $\leq n$  if and only if  $H^{i,j}(C) = 0$  for all  $i < j \leq n$  and  $i = 1, 2$ . The analogous statements are true for graded algebras and their homology.*

*Proof:* See [14]. □

**Proposition 2.** *A graded comodule  $P$  over a one-cogenerated coalgebra  $C$  is one-cogenerated if and only if  $H^{0,j}(C, P) = 0$  for all  $j > 0$ . A one-cogenerated comodule*



$P$  over a quadratic coalgebra  $C$  is quadratic iff  $H^{1,j}(C, P) = 0$  for all  $j > 1$ . More precisely, for any graded comodule  $P$  over a quadratic coalgebra  $C$  the morphism  $r_{C,P}: P \rightarrow q_C P$  is an isomorphism in degree  $\leq n$  if and only if  $H^{i,j}(C, P) = 0$  for all  $i < j \leq n$  and  $i = 0, 1$ . The analogous statements are true for graded modules.

*Proof:* The argument is based on the explicit form of the cobar-complex computing the cohomology in question. The space  $H^0(C, P)$  is isomorphic to the kernel of the map  $\Delta': P \rightarrow C_+ \otimes P$ ; if the coalgebra  $C$  is one-cogenerated, it follows easily that the morphism  $r_{C,P}: P \rightarrow q_C P \subset C \otimes P_0$  is injective in degree  $\leq n$  if and only if  $H^{0,j}(C, P) = 0$  for all  $0 < j \leq n$ . Now assume that the coalgebra  $C$  is quadratic and the map  $r_{C,P}$  is injective in degree  $< n$ . Let  $z \in C^+ \otimes P$  be a homogeneous cocycle of degree  $n$ ; then  $z = \sum_{s \geq 1, t \geq 0}^{s+t=n} z_{st}$  with  $z_{st} \in C_s \otimes P_t$ . The cocycle condition means that the images of  $(\Delta \otimes \text{id})(z_{u+v,w})$  and  $(\text{id} \otimes \Delta')(z_{u,v+w})$  in  $C_u \otimes C_v \otimes P_w$  coincide for any  $u, v \geq 1, w \geq 0, u + v + w = n$ . Since the comultiplication and coaction maps are injective, the latter equivalently means that the images of  $z_{st}$  in  $C_1^{\otimes n} \otimes P_0$  coincide for all  $s, t$ . So we get an element of  $C_1^{\otimes n} \otimes P_0$ ; it is easy to see that it represents an element of  $q_C P$ . Besides, this element belongs to the image of  $r_{C,P}$  iff the cocycle  $z$  is a coboundary. Finally, if the map  $r_{C,P}$  is an isomorphism in degree  $< n$ , then any element of  $q_C P$  of degree  $n$  corresponds to a cocycle  $z$  in this way.  $\square$

**Proposition 3.** For any graded coalgebra  $C$ , the diagonal subalgebra  $\bigoplus_i H^{i,i}(C)$  of the cohomology algebra  $H^*(C)$  is a quadratic algebra isomorphic to  $(qC)^!$ . For any graded  $C$ -comodule  $P$ , the diagonal part  $\bigoplus_i H^{i,i}(C, P)$  of the cohomology module  $H^*(C, P)$  is a quadratic module over the diagonal subalgebra of  $H^*(C)$  isomorphic to the  $(qC)^!$ -module  $(q_C P)_q^!$ . Analogously, for any graded algebra  $A$  the diagonal homology coalgebra  $\bigoplus_i H_{i,i}(A)$  is isomorphic to the quadratic coalgebra  $(qA)^?$ . For any graded  $A$ -module  $M$ , the diagonal homology  $\bigoplus_i H_{i,i}(A, M)$  is a quadratic comodule over the coalgebra  $\bigoplus_i H_{i,i}(A)$  isomorphic to the comodule  $(q_A M)_{qA}^?$  over  $(qA)^?$ .

*Proof:* Straightforward computation with the bar and cobar-complexes.  $\square$

**3.3. Koszul modules and comodules.** The notion of a *Koszul algebra* and the construction of the *Koszul complex* were invented by S. Priddy in [16]. The result relating Koszulity with distributivity of collections of vector subspaces is due to J. Backelin [1]. Detailed expositions can be found in the papers [5, 6]; see also [2].

A graded coalgebra  $C$  is called *Koszul* if one has  $H^{i,j}(C) = 0$  for all  $i \neq j$ . A graded comodule  $P$  over a Koszul coalgebra  $C$  is called *Koszul* if  $H^{i,j}(C, P) = 0$  for  $i \neq j$ . Analogously, a graded algebra  $A$  is called *Koszul* if  $H_{i,j}(A) = 0$  unless  $i = j$ . A graded module  $M$  over a Koszul algebra  $A$  is called *Koszul* if  $H^{i,j}(A, M) = 0$  unless  $i = j$ . Note that all Koszul algebras and coalgebras, as well as modules and comodules, are quadratic. This follows from Propositions 1 and 2.

It is clear from subsection 1.4 that algebras with opposite multiplication, as well as coalgebras with opposite comultiplication, are quadratic or Koszul simultaneously. A graded right module over a Koszul algebra  $A$  is called *Koszul* if it is a Koszul module over  $A^{\text{op}}$ ; analogously for right comodules.

Let  $A$  be a quadratic algebra and  $A^\sharp$  be the dual coalgebra; suppose we are given a right  $A$ -module  $M$  and a left  $A^\sharp$ -comodule  $P$ . The *Koszul complex*  $K(M, P)$  is defined as the vector space  $M \otimes P$  with the differential given by the formula

$$\partial = (m' \otimes \text{id})(\text{id} \otimes \Delta'): M \otimes P \longrightarrow M \otimes A_1 \otimes P \longrightarrow M \otimes P,$$

where  $m': M \otimes A_1 \longrightarrow M$  is the action morphism. It follows immediately from the definition of  $A^\sharp$  that one has  $\partial^2 = 0$ . For a left  $A$ -module  $M$  and a right  $A^\sharp$ -comodule  $P$ , the *Koszul complex*  $K(P, M)$  is defined in the analogous way.

**Proposition 4.** *A quadratic algebra  $A$  and the dual quadratic coalgebra  $A^\sharp$  are Koszul simultaneously. Moreover, they are Koszul if and only if the Koszul complex  $K(A, A^\sharp)$  is exact at all its terms  $A_p \otimes A_q^\sharp$  with  $p + q > 0$ . A quadratic module  $M$  over a Koszul algebra  $A$  and the dual quadratic comodule  $M_A^\sharp$  over the dual Koszul coalgebra  $A^\sharp$  are Koszul simultaneously, too. Furthermore, they are both Koszul if and only if the Koszul complex  $K(A, M_A^\sharp)$  is a resolution of the  $A$ -module  $M$  and if and only if the Koszul complex  $K(A^\sharp, M)$  is a resolution of the  $A^\sharp$ -comodule  $M_A^\sharp$ .*

*Proof:* Let us introduce a homological grading on the complex  $K(A, M_A^\sharp)$  by the rule  $K_i(A, M_A^\sharp) = A \otimes M_{A,i}^\sharp$ . There is a natural morphism of complexes of graded  $A$ -modules from  $K(A, M_A^\sharp)$  to the bar-resolution  $A \otimes M \longleftarrow A \otimes A_+ \otimes M \longleftarrow \cdots$  of the  $A$ -module  $M$  defined by the formula  $\rho(a \otimes p_i) = a \otimes \Delta'^{(i)}(p_i)$ , where  $\Delta'^{(i)}: M_{A,i}^\sharp \longrightarrow A_1^{\otimes i} \otimes M_0 \subset A_+^{\otimes i} \otimes M$  is the natural embedding given by the iterated coaction map. The composition of  $\rho$  with the standard morphism from the bar-resolution to the module  $M$  itself provides a morphism  $K(A, M_A^\sharp) \longrightarrow M$ .

Note that both the Koszul complex and the bar-resolution are complexes of free graded  $A$ -modules. It is not difficult to see (a version of Nakayama's lemma) that a morphism of complexes of free graded modules over an arbitrary graded algebra  $A$  is a quasi-isomorphism if and only if it becomes a quasi-isomorphism after tensoring with  $\mathbb{k}$  over  $A$ , provided that both complexes are concentrated in the positive homological degree. Tensoring with  $\mathbb{k}$  turns the bar-resolution to the bar-complex computing  $H_*(A, M)$  and the complex  $K(A, M)$  to the complex with the terms  $M_{A,i}^\sharp$  and zero differentials; now it follows from Proposition 3 that the map  $\mathbb{k} \otimes_A \rho$  is a quasi-isomorphism iff  $H_{i,j}(A, M) = 0$  for  $i \neq j$ .

Applying this result to the trivial  $A$ -module  $M = \mathbb{k}$ , we conclude that a quadratic algebra  $A$  is Koszul if and only if the Koszul complex  $K(A, A^\sharp)$  is a resolution of  $\mathbb{k}$ . Besides, we proved that a quadratic module  $M$  over a Koszul algebra  $A$  is Koszul iff the complex  $K(A, M_A^\sharp)$  is a resolution of the module  $M$ . The same arguments as above applied to a morphism of complexes of cofree graded comodules show that the comodule  $M_A^\sharp$  over  $A^\sharp$  is Koszul iff the complex  $K(A^\sharp, M)$  is a resolution of  $M_A^\sharp$ .

On the other hand, since the algebra  $A^{\text{op}}$  is Koszul, the complex  $K(A^\sharp, A)$  is a free graded resolution of the trivial right  $A$ -module. Computing the spaces  $\text{Tor}_*^A(\mathbb{k}, M)$  in terms of this resolution of  $\mathbb{k}$ , we get  $K(A^\sharp, A) \otimes_A M = K(A^\sharp, M)$  and  $H_*(A, M) = H_*K(A^\sharp, M)$ . So it follows from Proposition 3 that the map  $K(A^\sharp, M) \longrightarrow M_A^\sharp$  is an isomorphism on the homology iff the  $A$ -module  $M$  is Koszul. Now we see that the  $A$ -module  $M$  and the  $A^\sharp$ -comodule  $M_A^\sharp$  are Koszul simultaneously.  $\square$

**Remark:** The techniques of the above argument allow to figure out quite precisely the relations between the particular pieces of the Koszulity condition for the dual quadratic algebras and modules. For example, given two integers  $a, b \geq 2$  one has  $H_{i,j}(A) = 0$  for all  $i \leq a+1$  and  $0 < j-i \leq b-1$  if and only if  $H^{i,j}(A^?) = 0$  for all  $i \leq b+1$  and  $0 < j-i \leq a-1$ . Moreover, there are natural morphisms of vector spaces  $H^{b+1,a+b}(A^?) \rightarrow H_{a+1,a+b}(A)$ , which annihilate the nontrivial (co)multiplications on both sides. There are analogous results for dual quadratic modules.

**3.4. Koszulity and distributivity.** A collection of subspaces  $X_1, \dots, X_{n-1}$  in a vector space  $W$  is called *distributive* if there exists a (finite) direct decomposition  $W = \bigoplus_{\omega \in \Omega} W_\omega$  such that each subspace  $X_k$  is the sum of a set of subspaces  $W_\omega$ . Equivalently, a collection of subspaces  $X_k$  is distributive if the distributivity identity  $(X+Y) \cap Z = X \cap Z + Y \cap Z$  is satisfied for any triple of subspaces  $X, Y, Z$  which can be obtained from the subspaces  $X_k$  using the operations of sum and intersection.

**Proposition 5.** *The quadratic algebra  $A = \{V, R\}$  and the quadratic coalgebra  $C = \langle V, R \rangle$  are Koszul if and only if the collection of subspaces*

$$V^{\otimes k-1} \otimes R \otimes V^{n-k-1} \subset V^{\otimes n}, \quad k = 1, \dots, n-1$$

*is distributive for all  $n \geq 4$ . The quadratic module  $\{U, S\}_A$  over a Koszul algebra  $A = \{V, R\}$  and the quadratic comodule  $\langle U, S \rangle_C$  over the dual Koszul coalgebra  $C = \langle V, R \rangle$  are Koszul if and only if the collection of subspaces*

$$V^{\otimes k-1} \otimes R \otimes V^{n-k-1} \otimes U, \quad k = 1, \dots, n-1; \quad V^{\otimes n-1} \otimes S$$

*in the vector space  $V^{\otimes n} \otimes U$  is distributive for all  $n \geq 3$ .*

*Proof:* The first statement is proven in [14]; the proof of the second one is analogous and based on the same lemma.  $\square$

**Remark:** Many natural Koszulity-type homological conditions on data involving Koszul algebras and modules (including most of the properties that will be considered in section 5) can be simply rewritten in terms of distributivity of various collections of subspaces. It is not at all clear why this so happens. The related results often can be proven by both homological and lattice-theoretical methods, though the former are generally more powerful. Here are two of the various examples.

Let  $A = \{V, R\}$  be a Koszul algebra and  $E \subset V$  be a vector subspace. Then the subalgebra generated by  $E$  in the algebra  $A$  is Koszul if and only if the collection of subspaces  $E^{\otimes n}$  and  $V^{\otimes k-1} \otimes R \otimes V^{n-k-1}$  in the vector space  $V^{\otimes n}$  is distributive for all  $n \geq 3$ . Note that in general this subalgebra does not have to be quadratic.

Furthermore, let  $N = W \otimes A/TA$  and  $M = A \otimes U/AS$ , where  $T \subset W \otimes V$  and  $S \subset V \otimes U$ , be a left and a right Koszul  $A$ -modules. Then one has  $\text{Tor}_{i,n}^A(N, M) = 0$  for all  $i \neq 0, n$  if and only if the collection of subspaces  $W \otimes V^{\otimes k-1} \otimes R \otimes V^{n-k-1} \otimes U$  together with  $W \otimes V^{\otimes n-1} \otimes S$  and  $T \otimes V^{\otimes n-1} \otimes U$  is distributive in  $W \otimes V^{\otimes n} \otimes U$ . The latter observation was communicated to me by R. Bezrukavnikov.

## 4. CONILPOTENT COMODULES AND KOSZULITY

**4.1. Coaugmentation filtrations.** Let  $C$  be a coaugmented coalgebra with the coaugmentation map  $\gamma: \mathbb{k} \rightarrow C$ . Then the *coaugmentation filtration*  $N$  on the coalgebra  $C$  is an increasing filtration defined by the formula [14]

$$N_n C = \{c \in C \mid \Delta^{(n+1)}(c) \in C_\gamma^{\otimes n+1} = \sum_{i=1}^{n+1} C^{\otimes i-1} \otimes \gamma(\mathbb{k}) \otimes C^{\otimes n-i+1} \subset C^{\otimes n+1}\},$$

where  $\Delta^{(m)}: C \rightarrow C^{\otimes m}$  denotes the iterated comultiplication map. In particular, we have  $N_0 C = \gamma(\mathbb{k})$ .

Furthermore, let  $P$  be a comodule over a coaugmented coalgebra  $C$ . Then the *coaugmentation filtration*  $N$  on the comodule  $P$  is defined as

$$N_n P = \{p \in P \mid \Delta'(p) \in N_n C \otimes P \subset C \otimes P\}.$$

A coaugmented coalgebra  $C$  is called *conilpotent* if the coaugmentation filtration  $N$  is full, that is  $C = \bigcup_n N_n C$ . Note that in this case for any  $C$ -comodule  $P$  one has  $P = \bigcup_n N_n P$ ; in other words, *any comodule over a conilpotent coalgebra is conilpotent*.

For any coaugmented coalgebra  $C$ , we denote by  $\text{Nilp } C$  the maximal conilpotent subcoalgebra  $\bigcup_n N_n C$  of the coalgebra  $C$ . The following simple result is needed here.

**Proposition 6.** *The coaugmentation filtrations respect the coalgebra and comodule structures on a coalgebra  $C$  and comodule  $P$ , that is*

$$\Delta(N_n C) \subset \sum_{i+j=n} N_i C \otimes N_j C \quad \text{and} \quad \Delta'(N_n P) \subset \sum_{i+j=n} N_i C \otimes N_j P.$$

Furthermore, the associated graded coalgebra  $\text{gr}_N C = \bigoplus_{n=0}^{\infty} N_n C / N_{n-1} C$  and the comodule  $\text{gr}_N P = \bigoplus_{n=0}^{\infty} N_n P / N_{n-1} P$  over it are one-cogenerated.

*Proof:* The statements concerning the coalgebras only are proven in [14]. Let us show that for any comultiplicative filtration  $N$  on a coalgebra  $C$  the filtration on a  $C$ -comodule  $P$  given by the above formula is compatible with the coaction. Let  $\psi: C \rightarrow \mathbb{k}$  be a linear function annihilating  $N_{k-1} C$ ; we have to prove that  $(\psi \otimes \text{id})\Delta'(N_n P) \subset N_{n-k} P$ . The latter inclusion, by the definition, can be rewritten as  $\Delta'(\psi \otimes \text{id})\Delta'(N_n P) \subset N_{n-k} C \otimes P$ . Now we have

$$\begin{aligned} \Delta'(\psi \otimes \text{id})\Delta'(N_n P) &= (\psi \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta')\Delta'(N_n P) \\ &= (\psi \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})\Delta'(N_n P) \\ &\subset (\psi \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})(N_n C \otimes P) \subset N_{n-k} C \otimes P. \end{aligned}$$

Since one has  $\Delta'(p) \notin N_{n-1} C \otimes P$  for any  $p \notin N_{n-1} P$ , the last assertion of the proposition immediately follows.  $\square$

**4.2. Pro-finite groups.** Let  $G$  be a pro-finite group and  $\mathbb{k}$  be a field. Consider the coalgebra  $\mathbb{k}(G)$  of locally constant  $\mathbb{k}$ -valued functions on  $G$  with respect to the convolution; in other words, let  $\mathbb{k}(G) = \varinjlim \mathbb{k}(G/U)$ , where the direct limit is taken over all open normal subgroups  $U \subset G$  and the finite-dimensional coalgebra  $\mathbb{k}(G/U)$  is the dual vector space to the group algebra  $\mathbb{k}[G/U]$ . Then the category of discrete

$G$ -modules over  $\mathbb{k}$  is equivalent to the category of  $\mathbb{k}(G)$ -comodules. (Of course, the same is true over  $\mathbb{Z}$  or an arbitrary commutative coefficient ring.)

Now let  $\gamma : \mathbb{k} \rightarrow \mathbb{k}(G)$  be the coaugmentation map that takes a constant from  $\mathbb{k}$  to the corresponding constant function on  $G$ . Then for any discrete  $G$ -module  $P$  over  $\mathbb{k}$  one has  $H^*(G, P) = H^*(\mathbb{k}(G), P)$ , because  $\text{Ext}_G^*(\mathbb{k}, P) = \text{Ext}_{\mathbb{k}(G)}^*(\mathbb{k}, P)$ .

Suppose  $\mathbb{k}$  is a field of characteristic  $l$ . Then one can see that the coaugmented coalgebra  $\mathbb{k}(G)$  is conilpotent if and only if  $G$  is a pro- $l$ -group. Moreover, for any pro-finite group  $G$  the coalgebra  $\text{Nilp } \mathbb{k}(G)$  is isomorphic to the group coalgebra  $\mathbb{k}(G^{(l)})$  of the maximal quotient pro- $l$ -group  $G^{(l)}$  of  $G$ .

**4.3. Nilpotency and Koszulity.** The following theorem is the main result of my paper with A. Vishik [14].

**Theorem 3.** *Let  $A = H^*(C)$  be the cohomology algebra of a conilpotent coalgebra  $C$ . Assume that*

- (1) *the quadratic algebra  $qA$  is Koszul;*
- (2) *the morphism of graded algebras  $qA \rightarrow A$  is an isomorphism in degree 2 and a monomorphism in degree 3.*

*Then the algebra  $A$  is quadratic (and therefore, Koszul). In addition, the graded coalgebra  $\text{gr}_N C$  is Koszul and there is an isomorphism  $A \simeq (\text{gr}_N C)^!$ .  $\square$*

Our next theorem is a module version of the above one. Their proofs are completely parallel; still we prefer to give the full argument here.

**Theorem 4.** *Let  $C$  be a conilpotent coalgebra such that the cohomology algebra  $A = H^*(C)$  is Koszul (equivalently, a coalgebra  $C$  satisfies the conditions of Theorem 3). Let  $P$  be a comodule over  $C$ . Consider the cohomology module  $M = H^*(C, P)$  over the algebra  $A$ . Assume that*

- (1) *the quadratic  $A$ -module  $q_A M$  is Koszul;*
- (2) *the morphism of graded  $A$ -modules  $q_A M \rightarrow M$  is an isomorphism in degree 1 and a monomorphism in degree 2.*

*Then the  $A$ -module  $M$  is quadratic (and therefore, Koszul). In addition, the graded  $\text{gr}_N C$ -module  $\text{gr}_N P$  is Koszul and there is an isomorphism  $M \simeq (\text{gr}_N P)_{\text{gr}_N C}^!$  of graded modules over the graded algebra  $A \simeq (\text{gr}_N C)^!$ .*

*Proof:* The coaugmentation filtrations on the coalgebra  $C$  and comodule  $P$  induce multiplicative filtrations on their cobar-complexes by the standard rule

$$N_n C^{+\otimes i} = \sum_{j_1 + \dots + j_i = n} N_{j_1} C^+ \otimes \dots \otimes N_{j_i} C^+$$

and

$$N_n (C^{+\otimes i} \otimes P) = \sum_{j_1 + \dots + j_i + k = n} N_{j_1} C^+ \otimes \dots \otimes N_{j_i} C^+ \otimes N_k P,$$

where  $N_j C^+ = N_j C / \gamma(\mathbb{k})$ ; so the filtrations on the terms  $C^{+\otimes i}$  and  $C^{+\otimes i} \otimes P$  start with  $N_i$ . Clearly, the associated graded complexes coincide with the cobar-complexes

of  $\text{gr}_N C$  and  $\text{gr}_N P$ . Therefore, we obtain a multiplicative spectral sequence

$$E_1^{i,j} = H^{i,j}(\text{gr}_N C) \implies H^i(C)$$

and a module spectral sequence

$$'E_1^{i,j} = H^{i,j}(\text{gr}_N C, \text{gr}_N P) \implies H^i(C, P)$$

over it. Since the filtrations on the cobar-complexes are increasing ones, these spectral sequences converge in the direct limit. The differentials have the form  $d_r: E_r^{i,j} \longrightarrow E_r^{i+1,j-r}$  and  $'d_r: 'E_r^{i,j} \longrightarrow 'E_r^{i+1,j-r}$ . Furthermore, there are induced increasing filtrations  $N$  on  $H^*(C)$  and  $H^*(C, P)$  compatible with the multiplication and the action and such that  $\text{gr}_N^j H^i(C) = E_\infty^{i,j}$  and  $\text{gr}_N^j H^i(C, P) = 'E_\infty^{i,j}$ .

It was shown in the proof of Theorem 3 given in [14] that the spectral sequence  $E_r^{i,j}$  degenerates at  $E_1$  with  $E_1^{i,j} = E_\infty^{i,j} = 0$  for  $i \neq j$  and  $E_1^{i,i} = E_\infty^{i,i} = H^i(C) = A_i$ . In particular, we have  $N_i H^i(C) = H^i(C)$ . Besides, by Theorem 3 the graded algebra  $\text{gr}_N C$  is Koszul and  $H^*(\text{gr}_N C) \simeq (\text{gr}_N C)^! \simeq A$ .

From the spectral sequence  $'E_r^{i,j}$  we see that the submodule  $\bigoplus_{i=0}^\infty N_i H^i(C, P)$  of the  $A$ -module  $M = H^*(C, P)$  is isomorphic to the quotient module of the diagonal cohomology  $A$ -module  $\bigoplus_{i=0}^\infty H^{i,i}(\text{gr}_N C, \text{gr}_N P)$  by the images of the differentials. According to Proposition 6, the graded  $\text{gr}_N C$ -comodule  $\text{gr}_N P$  is one-cogenerated, hence (by Proposition 2) we have  $'E_1^{0,j} = H^{0,j}(\text{gr}_N P, \text{gr}_N C) = 0$  for  $j > 0$ , which implies  $H^0(C, P) = N_0 H^0(C, P) \simeq H^{0,0}(\text{gr}_N C, \text{gr}_N P)$ . Now the spectral sequence shows that  $E_\infty^{1,1} = E_1^{1,1}$  and therefore  $N_1 H^1(C, P) \simeq H^{1,1}(\text{gr}_N C, \text{gr}_N P)$ . Since (by Proposition 3) the diagonal cohomology  $A$ -module  $\bigoplus_{i=0}^\infty H^{i,i}(\text{gr}_N C, \text{gr}_N P)$  is quadratic, we conclude that it is isomorphic to  $q_A H^*(C, P)$ .

By Proposition 3 again, the  $A$ -module  $\bigoplus_{i=0}^\infty H^{i,i}(\text{gr}_N C, \text{gr}_N P)$  is quadratic dual to the  $\text{gr}_N C$ -comodule  $q_{\text{gr}_N C} \text{gr}_N P$ ; since we suppose that the  $A$ -module  $q_A H^*(C, P)$  is Koszul, the dual  $\text{gr}_N C$ -comodule  $q_{\text{gr}_N C} \text{gr}_N P$  is Koszul, too (Proposition 4). Besides, we have assumed that the morphism  $q_A H^*(C, P) \longrightarrow H^*(C, P)$  is an isomorphism in degree 1, hence  $H^1(C, P) = N_1 H^1(C, P)$  and  $'E_\infty^{1,n} = 0$ . Furthermore, we have supposed that the morphism  $q_A H^*(C, P) \longrightarrow H^*(C, P)$  is a monomorphism in degree 2. Since  $'E_1^{2,2} = H^{2,2}(\text{gr}_N C, \text{gr}_N P) \simeq q_A H^*(C, P)_2$  and  $'E_\infty^{2,2} \simeq N_2 H^2(C, P)$ , it follows that the map  $'E_1^{2,2} \longrightarrow 'E_\infty^{2,2}$  is a monomorphism and therefore all the differentials  $'d_r: 'E_r^{1,2+r} \longrightarrow 'E_r^{2,2}$  targeting at  $'E_r^{2,2}$  vanish.

Now let us establish by induction on  $j$  that  $H^{1,j}(\text{gr}_N C, \text{gr}_N P) = 0$  for all  $j > 1$ . Assume that this is true for  $1 < j \leq n-1$ . By Proposition 2, it follows that the map  $r_{\text{gr}_N C, \text{gr}_N P}: \text{gr}_N P \longrightarrow q_{\text{gr}_N C} \text{gr}_N P$  is an isomorphism in degree  $\leq n-1$ . Therefore, the induced map of the cobar-complexes is also an isomorphism in these degrees, hence in particular  $H^{1,j}(\text{gr}_N C, \text{gr}_N P) = H^{1,j}(\text{gr}_N C, q_{\text{gr}_N C} \text{gr}_N P)$  for  $j \leq n-1$  (and actually even for  $j \leq n$ ). Since the  $\text{gr}_N C$ -comodule  $q_{\text{gr}_N C} \text{gr}_N P$  is Koszul, it follows that  $'E_1^{2,j} = H^{2,j}(\text{gr}_N C, \text{gr}_N P) = 0$  for all  $2 < j \leq n-1$ . From the latter it is clear that  $'E_1^{1,n} = 'E_\infty^{1,n}$ , since the differentials targeting at  $'E_r^{2,2}$  also vanish. It was noticed above that  $'E_\infty^{1,n} = 0$ ; so  $H^{1,n}(\text{gr}_N C, \text{gr}_N P) = 'E_1^{1,n} = 0$  and we are done.

We have shown that the  $\mathrm{gr}_N C$ -comodule  $\mathrm{gr}_N P$  is quadratic and the comodule  $q_{\mathrm{gr}_N C} \mathrm{gr}_N P$  is Koszul, hence the comodule  $\mathrm{gr}_N P$  is Koszul. It follows that  $'E_1^{i,j} = 0$  for  $i \neq j$ , so the spectral sequence degenerates and  $H^*(C, P) = H^*(\mathrm{gr}_N C, \mathrm{gr}_N P) = (\mathrm{gr}_N P)_{\mathrm{gr}_N C}^!$ . Therefore,  $H^*(C, P)$  is a Koszul  $H^*(C)$ -module, too.  $\square$

## 5. MAXIMAL CONILPOTENT SUBCOALGEBRA AND KOSZULITY

The results of this section connect the cohomology of a coaugmented coalgebra with that of its maximal conilpotent subcoalgebra (and, in particular, the cohomology of a pro-finite group with the cohomology of its maximal quotient pro- $l$ -group) under certain Koszulity conditions. Let us start with the following important lemma.

**Lemma 1.** *Let  $C$  be a coaugmented coalgebra and  $\mathrm{Nilp} C$  be its maximal conilpotent subcoalgebra; then the natural map  $H^*(\mathrm{Nilp} C) \rightarrow H^*(C)$  is an isomorphism in degree 1 and a monomorphism in degree 2. In particular, the quadratic algebras  $qH^*(C)$  and  $qH^*(\mathrm{Nilp} C)$  are naturally isomorphic to each other.*

*Proof:* Any comodule over the coalgebra  $\mathrm{Nilp} C$  can be considered as a comodule over  $C$  in the most obvious way. Furthermore, a  $C$ -comodule comes from a comodule over  $\mathrm{Nilp} C$  if and only if it is a direct limit of successive extensions of the trivial  $C$ -comodule  $\mathbb{k}$  (which is defined in terms of the coaugmentation). This is not difficult to prove, in the “if” direction by induction, using the coaugmentation filtrations on comodules and Proposition 6. It is helpful to remember that every comodule over a coassociative coalgebra is a direct limit of finite-dimensional comodules.

For any two comodules  $P$  and  $Q$  over the subcoalgebra  $\mathrm{Nilp} C$  there is a natural morphism of graded vector spaces  $\mathrm{Ext}_{\mathrm{Nilp} C}^*(P, Q) \rightarrow \mathrm{Ext}_C^*(P, Q)$ . Now it follows from the above that this map is always an isomorphism in degree 1. It turns out that one can derive purely formally from the latter that this is also a monomorphism in degree 2. Namely, it suffices to compute the spaces  $\mathrm{Ext}^2$  in question in terms of  $\mathrm{Ext}^1$  using a one-step injective resolution of the comodule  $Q$  over the coalgebra  $\mathrm{Nilp} C$ , for example  $Q \rightarrow \mathrm{Nilp} C \otimes Q \rightarrow (\mathrm{Nilp} C \otimes Q)/Q$ .  $\square$

The next result can be considered as a direct generalization of Theorem 3.

**Theorem 5.** *Let  $C$  be a coaugmented coalgebra such that the cohomology algebra  $H^*(C)$  satisfies the conditions of Theorem 3. Then the cohomology algebra of the maximal conilpotent subcoalgebra  $\mathrm{Nilp} C$  is isomorphic to the quadratic part of the cohomology of  $C$ , that is  $H^*(\mathrm{Nilp} C) \simeq qH^*(C)$ .*

*Proof:* Let us show that the cohomology algebra of the coalgebra  $\mathrm{Nilp} C$  satisfies the conditions of Theorem 3, too. Indeed, the second statement of Lemma 1 provides the condition (1). To check the condition (2), one should consider the composition  $qH^*(C) \simeq qH^*(\mathrm{Nilp} C) \rightarrow H^*(\mathrm{Nilp} C) \rightarrow H^*(C)$  and use the first statement of Lemma 1. Applying Theorem 3, we conclude that the algebra  $H^*(\mathrm{Nilp} C)$  is quadratic. The latter implies the desired isomorphism of graded algebras.  $\square$

**Corollary 1.** *Let  $C$  be a coaugmented coalgebra and  $\text{Nilp } C$  be its maximal conilpotent subcoalgebra. In this setting, if the cohomology algebra  $H^*(C)$  is Koszul, then the natural homomorphism  $H^*(\text{Nilp } C) \longrightarrow H^*(C)$  is an isomorphism.*  $\square$

The analogous results comparing the cohomology of a pro-finite group with that of its maximal quotient pro- $l$ -group can be either deduced from the results for coalgebras using the observations made in subsection 3.2, or they can be proven independently in the very parallel way. The proof of the analogue of Lemma 1 can be done differently (and even simpler) for pro-finite groups.

**Lemma 2.** *If  $G$  is a pro-finite group and  $G^{(l)}$  is its maximal quotient pro- $l$ -group, then the natural map  $H^i(G^{(l)}, \mathbb{Z}/l) \longrightarrow H^i(G, \mathbb{Z}/l)$  is an isomorphism for  $i = 1$  and a monomorphism for  $i = 2$ . In particular, the quadratic algebras  $\text{q}H^*(G, \mathbb{Z}/l)$  and  $\text{q}H^*(G^{(l)}, \mathbb{Z}/l)$  are naturally isomorphic to each other.*

*Proof:* Let  $G^{(l)} = G/G'$ ; then by the definition the subgroup  $G'$  has no nontrivial quotient pro- $l$ -groups, so  $H^1(G', \mathbb{Z}/l) = 0$ . It remains to apply the Serre–Hochschild spectral sequence  $E_2^{p,q} = H^p(G^{(l)}, H^q(G', \mathbb{Z}/l)) \implies H^{p+q}(G, \mathbb{Z}/l)$ .  $\square$

**Corollary 2.** *Let  $G$  be a pro-finite group and  $G^{(l)}$  be its maximal quotient pro- $l$ -group. If the cohomology algebra  $H^*(G, \mathbb{Z}/l)$  satisfies the conditions of Theorem 3, then the cohomology algebra of the group  $G^{(l)}$  is isomorphic to the quadratic part of the cohomology of  $G$ ,*

$$H^*(G^{(l)}, \mathbb{Z}/l) \simeq \text{q}H^*(G, \mathbb{Z}/l).$$

Furthermore, if the cohomology algebra  $H^*(G, \mathbb{Z}/l)$  is Koszul, then the natural homomorphism

$$H^*(G^{(l)}, \mathbb{Z}/l) \longrightarrow H^*(G, \mathbb{Z}/l)$$

is an isomorphism.  $\square$

**Remark:** All the results of this section actually have a rather general categorical nature: they can be formulated for an arbitrary abelian category (or, better yet, a triangulated category) in place of the category of  $C$ -comodules with a “nilpotent” abelian subcategory in place of the subcategory of comodules over  $\text{Nilp } C$ ; the nilpotency can be generalized to mean that all the objects have finite length. The proof is based on the generalization of Theorem 3 to nilpotent abelian categories. We wouldn’t go so far here, but will only mention several elementary applications.

Let  $A$  be an augmented algebra. The *coalgebra of pro-nilpotent completion*  $C = A^\wedge$  of the algebra  $A$  is defined as  $C = \varinjlim_I (A/I)^*$ , where the limit goes over all the ideals  $I \subset A_+$  such that the algebra  $A/I$  is a finite-dimensional nilpotent algebra (meaning that there exists such  $n$  that  $(A_+/I)^n = 0$ ) and  $(A/I)^*$  is the dual vector space to  $A/I$  with its natural coalgebra structure. Then the following analogue of Theorem 5 is true: if the cohomology algebra  $H^*(A) = \text{Ext}_A^*(\mathbb{k}, \mathbb{k})$  satisfies the conditions of Theorem 3, then the algebra  $H^*(C)$  is isomorphic to the quadratic part of  $H^*(A)$ .

Furthermore, the analogue of Corollary 2 is valid for a discrete group  $\Gamma$  in place of  $G$  and its pro- $l$ -completion  $\Gamma^{(l)}$  or pro-unipotent completion  $\Gamma^{(\mathbb{Q})}$  in place of  $G^{(l)}$ , where one considers the cohomology with the constant coefficients  $\mathbb{Z}/l$  or  $\mathbb{Q}$ , respectively.



One can reduce the last two statements to the one about the cohomology of an augmented algebra using the results of Quillen's paper [17], or prove all the three assertions independently just in the same way as it was done for coalgebras above.

## 6. MORPHISMS OF GRADED ALGEBRAS AND KOSZULITY

The results of this and the next sections were found in an attempt to generalize and clarify some of the statements from the paper of J. Backelin and R. Fröberg [2].

Suppose we are given a homomorphism of algebras  $f: A \longrightarrow B$ . Then there is an induced structure of left  $A$ -module on the vector space  $B$ . For any right  $A$ -module  $M$  and left  $B$ -module  $N$  there is a natural isomorphism  $(M \otimes_A B) \otimes_B N \simeq M \otimes_A N$ , which implies the analogous isomorphism on the level of derived functors. Thus we obtain a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_p^B(\mathrm{Tor}_q^A(M, B), N) \implies \mathrm{Tor}_{p+q}^A(M, N)$$

with the differentials  $d^r: E_{p,q}^r \longrightarrow E_{p-r, q+r-1}^r$ . In particular, for a homomorphism of augmented algebras  $f$  and the trivial modules  $M = \mathbb{k}$  and  $N = \mathbb{k}$  we get

$$E_{p,q}^2 = H_p(B^{\mathrm{op}}, H_q(A, B)) \implies H_{p+q}(A).$$

Recall that in our notation  $\mathrm{Tor}_*^A(M, \mathbb{k}) = H_*(A^{\mathrm{op}}, M)$  and  $\mathrm{Tor}_*^B(\mathbb{k}, N) = H_*(B, N)$ ; besides,  $H_*(A) = H_*(A^{\mathrm{op}}) = \mathrm{Tor}_*^A(\mathbb{k}, \mathbb{k})$ .

If  $A$  and  $B$  are graded algebras ( $M$  and  $N$  are graded modules) and the homomorphism  $f$  preserves the grading, then all the terms of the spectral sequence  $E_{p,q}^r$  bear the corresponding additional grading which is preserved by all the differentials.

We will be interested in two specific cases, which can be thought of as *two kinds of Koszulity condition on a morphism*  $f: A \longrightarrow B$ . The first case is when one has  $H_{i,j}(A, B) = 0$  unless  $i = j$ . A bit more general, but nevertheless interesting, second case is when  $H_{i,j}(A, B) = 0$  unless  $j - i \leq 1$ . Note that the first condition implies surjectivity of the morphism  $f$ ; the second one doesn't.

For a nonnegatively graded vector space  $N = \bigoplus_{i=0}^{\infty} N_i$  and an integer  $n \geq 0$  let us denote by  $N(n)$  the graded vector space with the components  $N(n)_i = N_{i-n}$ . We will say that a graded module  $M$  over a Koszul algebra  $A$  is a *Koszul module in the grading shifted by  $n$*  if there exists a Koszul  $A$ -module  $N$  such that  $M = N(n)$ .

**Theorem 6.** *Let  $f: A \longrightarrow B$  be a homomorphism of graded algebras such that the corresponding vector spaces  $H_{i,j}(A, B)$  are zero unless  $j - i \leq 1$ . Then if the graded algebra  $A$  is Koszul, then the algebra  $B$  is Koszul also. Furthermore, if  $H_{i,j}(A, B) = 0$  unless  $i = j$ , then the graded algebras  $A$  and  $B$  are Koszul simultaneously.*

*Proof:* We have to prove that  $H_{p,j}(B) = 0$  for all  $j \neq p$ . Proceeding by induction, assume that this is true for all  $0 \leq p < n$ . Since  $H_{q,j}(A, B) = 0$  for all  $q \neq j, j+1$  and the graded right  $B$ -module  $H_q(A, B)$  can be presented as an extension of its grading components endowed with the trivial  $B$ -module structures, it follows immediately that for any  $p < n$  the term  $E_{p,q}^2 = H_p(B^{\mathrm{op}}, H_q(A, B))$  of the above spectral sequence has no components of degrees other than  $p+q$  and  $p+q+1$ .

Now let us consider the term  $E_{n,0}^2$ . The graded components of the terms  $E_{n,0}^r$  can only cancel in the spectral sequence with those of the terms  $E_{n-r,r-1}^r$ ; according to the above, the latter are concentrated in the degrees  $n-1$  and  $n$ . The limit term  $E_{n,0}^\infty = \text{gr}_n H_n(A)$  is known to be only nonzero in degree  $n$ ; it follows easily that the graded vector space  $H_n(B, H_0(A, B)) = E_{n,0}^2$  is concentrated in degree  $n$ .

By assumption, the graded vector space  $H_0(A, B) = B/f(A)$  is concentrated in the degrees 0 and 1. Thus we have an exact triple of graded right  $B$ -modules

$$0 \longrightarrow B_1/f(A_1)(1) \longrightarrow H_0(A, B) \longrightarrow \mathbb{k} \longrightarrow 0,$$

where the vector spaces  $B_1/f(A_1)$  and  $\mathbb{k}$  have the trivial  $B$ -module structures. Consider the corresponding long exact sequence of homology: the term  $H_n(B)$  is placed between  $H_n(B^{\text{op}}, H_0(A, B))$  and  $H_{n-1}(B) \otimes B_1/f(A_1)(1)$  and so it follows from the induction hypothesis that  $H_{n,j}(B) = 0$  for all  $j \neq n$ .

Conversely, if the graded spaces  $H_p(B)$  are concentrated in degree  $p$  and the spaces  $H_q(A, B)$  are concentrated in degree  $q$ , then all the terms  $E_{p,q}^2 = H_p(B^{\text{op}}, H_q(A, B))$  are concentrated in degree  $p+q$ . In this case the same holds for the terms  $E_{p,q}^\infty = \text{gr}_p H_{p+q}(A)$ , hence the spaces  $H_n(A)$  are concentrated in degree  $n$ .  $\square$

The next corollary lists the most typical specific situations (see also [2]).

**Corollary 3.** *Let  $f: A \longrightarrow B$  be a homomorphism of graded algebras. Suppose that the algebra  $A$  is Koszul and one of the following three conditions holds:*

- (a) *the  $A$ -module  $B$  is Koszul (equivalently, the morphism  $f$  is surjective and its kernel  $J$  is a Koszul  $A$ -module in the grading shifted by 1), or*
- (b) *the morphism  $f$  is injective and its cokernel  $B/f(A)$  is a Koszul  $A$ -module in the grading shifted by 1, or*
- (c) *the morphism  $f$  is surjective and its kernel  $J$  is a Koszul  $A$ -module in the grading shifted by 2.*

*Then the algebra  $B$  is Koszul, too.*  $\square$

The proof of the following result is completely analogous to that of Theorem 6.

**Proposition 7.** *Let  $f: A \longrightarrow B$  be a morphism of graded algebras. Assume that  $H_{0,j}(A, B) = 0$  for all  $j \neq 0, 1$  and  $H_{i',j}(B^{\text{op}}, H_{i''}(A, B)) = 0$  for all  $i'' \geq 1$  and  $j - i' - i'' \neq 0, 1$ . In this case if the algebra  $A$  is Koszul, then the algebra  $B$  is Koszul, too. If the algebra  $B$  is Koszul and all the right  $B$ -modules  $H_i(A, B)$  are Koszul in the grading shifted by  $i$ , then the algebra  $A$  is Koszul.*  $\square$

**Corollary 4.** *Let  $f: A \longrightarrow B$  be a homomorphism of graded algebras. Assume that  $A$  is a Koszul algebra and one of the following three conditions holds:*

- (a) *the algebra  $B$  is a free left  $A$ -module and the right  $B$ -module  $H_0(A, B) = B/A_+B$  is Koszul, or*
- (b) *the morphism  $f$  is surjective, its kernel  $J$  is a free left  $A$ -module, and the right  $B$ -module  $H_0(A, J) = H_1(A, B)$  is Koszul in the grading shifted by 1, or*
- (c) *same as (b), except that the module is Koszul in the grading shifted by 2.*

Then the algebra  $B$  is Koszul, too.  $\square$

**Proposition 8.** *Let  $f: A \rightarrow B$  be a morphism of Koszul algebras. In this case if the algebra  $B$  is a free left  $A$ -module, then the right  $B$ -module  $H_0(A, B)$  is Koszul. If the morphism  $f$  is surjective and its kernel  $J$  is a free left  $A$ -module, then one has  $H_{i,j}(B^{\text{op}}, H_0(A, J)) = 0$  for all  $j - i \neq 1, 2$ . More specifically, the right  $B$ -module  $H_0(A, J)$  is Koszul in the grading shifted by 1 iff the morphism of dual coalgebras  $f^?: H_*(A) \rightarrow H_*(B)$  is surjective; the module  $H_0(A, J)$  is Koszul in the grading shifted by 2 iff the morphism  $f^?$  is injective.*

*Proof:* All the statements follow from the spectral sequence of the morphism  $f$ .  $\square$

## 7. MORPHISMS OF KOSZUL ALGEBRAS: KOSZULITY AND FREENESS

Let  $f: A \rightarrow B$  be a homomorphism of Koszul algebras. Note that just as the morphism  $f$  makes the algebra  $B$  a left and a right  $A$ -module, the dual morphism  $f^?: A^? \rightarrow B^?$  makes the coalgebra  $A^?$  a left and a right  $B^?$ -comodule. In this section we describe the correspondence between the homological conditions of the above type on the morphisms  $f$  and  $f^?$ . Here is the principal result.

**Theorem 7.** *Let  $f: A \rightarrow B$  be a morphism of Koszul algebras and  $f^?: A^? \rightarrow B^?$  be the dual morphism of Koszul coalgebras. Then there are natural isomorphisms of the (co)module (co)homology vector spaces*

$$H_{i,j}(A, B) \simeq H^{j-i,j}(B^{\text{op}}, A^{\text{op}})$$

*compatible with the right action of  $B$  and the left coaction of  $A^?$  on both sides.*

*Proof:* According to Proposition 4, the Koszul complex  $K(A^?, A)$  with the homological grading  $K_i(A^?, A) = A_i^? \otimes A$  is a free graded right  $A$ -module resolution of the trivial  $A$ -module  $\mathbb{k}$ . Therefore, the space  $H_{i,j}(A, B)$  can be computed as the homology space of the complex  $K(A^?, A) \otimes_A B = A^? \otimes B$  at the term  $A_i^? \otimes B_{j-i}$ . Analogously, the Koszul complex  $K(B^?, B)$  with the cohomological grading  $K^i(B^?, B) = B^? \otimes B_i$  is a cofree graded left  $B^?$ -comodule resolution of the trivial  $B^?$ -comodule  $\mathbb{k}$ . It follows that the space  $H^{i,j}(B^{\text{op}}, A^{\text{op}})$  is isomorphic to the cohomology space of the complex  $A^? \odot_{B^?} K(B^?, B) = A^? \otimes B$  at the term  $A_{j-i}^? \otimes B_i$ . Now we see that both sides of the desired isomorphism of vector spaces are computed by the Koszul complex  $K(f; A^?, B) = A^? \otimes B$  with the differential given by the formula

$$\begin{aligned} \partial &= (\text{id} \otimes m)(\text{id} \otimes f_1 \otimes \text{id})(\Delta \otimes \text{id}): A^? \otimes B \longrightarrow A^? \otimes A_1 \otimes B \\ &\longrightarrow A^? \otimes B_1 \otimes B \longrightarrow A^? \otimes B. \end{aligned}$$

The right action of  $B$  and the left coaction of  $A^?$  on both  $H_*(A, B)$  and  $H^*(B^{\text{op}}, A^{\text{op}})$  come from the natural action and coaction on this Koszul complex.  $\square$

**Corollary 5.** *The conditions (a), (b), (c) of Corollaries 3 and 4 are respectively dual to each other in the following sense. Let  $f: A \rightarrow B$  be a morphism of Koszul algebras and  $f^?: A^? \rightarrow B^?$  be the dual morphism of Koszul coalgebras. Then the*

morphism  $f$  satisfies the right module version of a condition from Corollary 3 if and only if the morphism  $f^?$  satisfies the coalgebra version of the corresponding condition from Corollary 4, and vice versa. Specifically, the following statements hold.

- (a) The algebra  $B$  is a free right  $A$ -module if and only if the coalgebra  $A^?$  is a Koszul left  $B^?$ -comodule. If this is the case, the left  $B$ -module  $H_0(A^{\text{op}}, B)$  is the Koszul module dual to the left  $B$ -comodule  $A^?$ .
- (b-c) If the morphism  $f$  is surjective, then its kernel  $J$  is a free right  $A$ -module if and only if  $H^{i,j}(B^?, A^?) = 0$  for all  $j - i \neq 0, 1$ . (One has  $H_{i,i}(B^?, A^?) = 0$  for all  $i > 0$  and  $H_{j-1,j}(B^?, A^?) \simeq H_{0,j}(A^{\text{op}}, J)$  for all  $j$ .)

In the second case, if (b) the morphism  $f^?$  is surjective, then its kernel  $K$  and the space  $H_0(A^{\text{op}}, J)$  of graded generators of the right ideal  $J$  are dual left Koszul  $B^?$ -comodule and  $B$ -module, both in the grading shifted by 1. If (c) the map  $f_1$  is an isomorphism, then the cokernel  $C$  of the morphism  $f$  and the generator space  $H_0(A^{\text{op}}, J)$  are dual left Koszul  $B^?$ -comodule and  $B$ -module, both in the grading shifted by 2.  $\square$

## 8. GRADED LIE COALGEBRAS AND KOSZULITY

It turns out somewhat unexpectedly that the results Theorem 7 implies for morphisms of Koszul commutative and Lie (co)algebras are stronger than in general. On the other hand, Lie coalgebras involve some troubles [11], which disappear in the positively graded case.

By a *graded Lie algebra* we mean here a positively graded vector space  $L = \bigoplus_{i=1}^{\infty} L_n$  equipped with a Lie algebra structure such that  $[L_i, L_j] \subset L_{i+j}$  for all  $i, j$ . The universal enveloping algebra  $U(L)$  of a graded Lie algebra is a graded associative algebra. A graded Lie algebra is called *quadratic* if it is isomorphic to the quotient algebra of the free Lie algebra generated by  $L_1$  by an ideal generated by a set of elements of degree 2. A graded Lie algebra  $L$  is quadratic if and only if the algebra  $U(L)$  is quadratic. The homology coalgebra of a Lie algebra  $L$  can be defined as the homology of its enveloping algebra,  $H_*(L) = H_*(U(L))$ , or, more explicitly, as the homology coalgebra of the standard complex

$$H_*(L) = H_*(\mathbb{k} \longleftarrow L \longleftarrow \bigwedge^2 L \longleftarrow \bigwedge^3 L \longleftarrow \cdots).$$

From the second definition it is obvious that the homology coalgebra of a Lie algebra is always skew-cocommutative. By a *graded super-Lie algebra* we mean a positively graded vector space equipped with a super-Lie algebra structure with respect to the parity induced by the grading such that the bracket respects the grading. There is a similar standard complex computing the homology of a super-Lie algebra.

**Proposition 9.** *A quadratic algebra  $A$  is isomorphic to the enveloping algebra of a quadratic Lie algebra if and only if the dual quadratic coalgebra  $A^?$  is skew-cocommutative. A quadratic algebra  $A$  is isomorphic to the enveloping algebra of a quadratic super-Lie algebra iff the coalgebra  $A^?$  is cocommutative.  $\square$*

A graded Lie algebra or super-Lie algebra  $L$  is called *Koszul* if one has  $H_{i,j}(L) = 0$  for  $i \neq j$ ; in other words,  $L$  is Koszul if the enveloping associative algebra  $U(L)$  is Koszul. Let us denote by  $L^? = U(L)^?$  the quadratic coalgebra dual to  $U(L)$ .

**Corollary 6.** *Let  $\phi: L' \rightarrow L''$  be a morphism of Koszul Lie algebras or super-Lie algebras. Then the coalgebra  $L'^?$  is a Koszul  $L''^?$ -comodule if and only if the morphism  $\phi$  is injective. Furthermore, one has  $H^{i,j}(L''^?, L'^?) = 0$  for all  $j - i \neq 0, 1$  if and only if the kernel  $L$  of the morphism  $\phi$  is a free graded (super)-Lie algebra.*

*Proof:* It follows from the Poincaré–Birkhoff–Witt theorem that the enveloping algebra  $U(L'')$  is a free graded  $U(L)$ -module if and only if the morphism  $\phi$  is injective, so it remains to apply Corollary 5 (a) to prove the first statement. For an arbitrary morphism of Lie algebras  $\phi$ , it is clear that the algebra  $U(L'')$  is a free module over the algebra  $U(\text{im } \phi) = U(L'/L)$ , the  $U(L')$ -module  $U(L'/L)$  is induced from the trivial  $U(L)$ -module, and thus there are natural isomorphisms  $H_*(U(L'), U(L'')) \simeq H_*(U(L'), U(L'/L)) \otimes_{U(L'/L)} U(L'') \simeq H_*(L) \otimes_{U(L'/L)} U(L'')$ . Therefore, one has  $H_i(U(L'), U(L'')) = 0$  for  $i \geq 2$  if and only if  $L$  is a free Lie algebra and the second statement follows from Theorem 7.  $\square$

Recall that a *Lie coalgebra* is a vector space  $\Lambda$  together with a cobracket map  $\delta: \Lambda \rightarrow \bigwedge^2 \Lambda$  satisfying the equation dual to the Jacobi identity. The cohomology algebra of a Lie coalgebra  $\Lambda$  is defined as the cohomology of the standard complex

$$H^*(\Lambda) = H^*(\mathbb{k} \rightarrow \Lambda \rightarrow \bigwedge^2 \Lambda \rightarrow \bigwedge^3 \Lambda \rightarrow \dots).$$

The notion of the coenveloping coalgebra of a Lie coalgebra involves troubles, which may be overcome by restricting to the following condition. A Lie coalgebra is called *conilpotent* if it is a direct limit of finite-dimensional Lie coalgebras dual to nilpotent Lie algebras. The *conilpotent enveloping coalgebra*  $C(\Lambda)$  of a conilpotent Lie coalgebra  $\Lambda$  is defined as the universal object in the category of morphisms of conilpotent Lie coalgebras  $C \rightarrow \Lambda$  from the Lie coalgebra associated with a conilpotent coassociative coalgebra  $C$  to the Lie coalgebra  $\Lambda$ . One can show that for any conilpotent Lie coalgebra  $\Lambda$  the cohomology algebras  $H^*(\Lambda)$  and  $H^*(C(\Lambda))$  are naturally isomorphic.

By a *graded Lie coalgebra* we mean a positively graded vector space  $\Lambda = \bigoplus_{n=1}^{\infty} \Lambda_n$  endowed with a Lie coalgebra structure such that  $\delta(\Lambda_n) \subset \sum_{i+j=n} \Lambda_i \wedge \Lambda_j$ . Obviously, any positively graded Lie coalgebra is conilpotent. The conilpotent coenveloping coalgebra of a graded Lie coalgebra is naturally a graded coassociative coalgebra and coincides with the coenveloping coalgebra defined with respect to the category of graded coalgebras. There are analogous notions of a *conilpotent super-Lie coalgebra*, its coenveloping coalgebra, and its cohomology algebra. A graded (super)-Lie coalgebra  $\Lambda$  is called *Koszul* if one has  $H^{i,j}(\Lambda) = 0$  for all  $i \neq j$ .

**Corollary 7.** *Let  $f: A \rightarrow B$  be a morphism of commutative or skew-commutative Koszul algebras. Then the algebra  $B$  is a Koszul  $A$ -module if and only if the dual morphism  $f^?: A^? \rightarrow B^?$  is surjective.*

*Proof:* This follows from the appropriate version of Proposition 9 and the analogue of Corollary 6 for morphisms of Koszul Lie or super-Lie coalgebras. An important difference is that the computation with enveloping algebras from the proof of the latter Corollary wouldn't work for coenveloping coalgebras in general, since the dual version of Poincaré–Birkhoff–Witt theorem does not always hold. It holds for conilpotent Lie coalgebras, though.  $\square$

By the definition, a *pro-Lie algebra* is a projective limit of finite-dimensional Lie algebras and a *pro-nilpotent Lie algebra* is a projective limit of finite-dimensional nilpotent Lie algebras. Note that the category of pro-Lie algebras is anti-equivalent to the category of locally finite Lie coalgebras and the category of pro-nilpotent Lie algebras is anti-equivalent to the category of conilpotent Lie coalgebras. Any positively graded pro-Lie algebra is, of course, pro-nilpotent. The cohomology algebra of a pro-nilpotent Lie algebra is defined as the cohomology of the corresponding Lie coalgebra; a graded pro-nilpotent (super)-Lie algebra is called Koszul if the corresponding (super)-Lie coalgebra is Koszul. A graded pro-nilpotent (super)-Lie algebra is free iff the corresponding (super)-Lie coalgebra is conilpotent cofree.

**Corollary 8.** *Let  $\phi: \mathfrak{L}' \rightarrow \mathfrak{L}''$  be a morphism of Koszul pro-Lie algebras or pro-super-Lie algebras. Then one has  $H_{i,j}(H^*(\mathfrak{L}''), H^*(\mathfrak{L}')) = 0$  for all  $j - i \neq 0, 1$  if and only if the kernel  $\mathfrak{L}$  of the morphism  $\phi$  is a free pro-(super)-Lie algebra.*

This is the dual version of the second statement of Corollary 6.  $\square$

The result about freeness of a graded pro-Lie algebra mentioned in subsection 1.8 of the Introduction now follows immediately from Theorem 6 and Corollary 8.

## 9. MAIN THEOREM

Two different proofs of Theorem 2 are given below, one based on Theorem 4 and the other on Theorem 7. Both approaches actually lead to somewhat stronger results, each in its own direction; they are formulated here as Theorem 9 and Corollary 9.

**Theorem 8.** *Let  $g: C \rightarrow D$  be a morphism of conilpotent coalgebras. Assume that the cohomology algebras  $A = H^*(C)$  and  $B = H^*(D)$  are Koszul. In this case, if for a certain integer  $t \geq 1$  one has  $H_{i,j}(A^{\text{op}}, B^{\text{op}}) = 0$  for all  $j - i \geq t$  (or even only for all  $j - i = t$ ), then  $H^i(D, C) = 0$  for all  $i \geq t$ . For example, if the right  $A$ -module  $B$  is Koszul, then the left  $D$ -comodule  $C$  is cofree. On the other hand, if the algebra  $B$  is Koszul, then the morphism  $C \rightarrow D$  is injective and at the same time the  $B$ -module  $H^*(D, C)$  is Koszul if and only if the algebra  $A$  is Koszul and at the same time the right  $A$ -module  $B$  is free.*

*Proof:* Note that any morphism of conilpotent coalgebras  $g: C \rightarrow D$  commutes with the coaugmentations. For the coaugmentation filtrations  $N$  on the coalgebras  $C$  and  $D$  one obviously has  $g(N_i C) \subset N_i D$  and it follows that these filtrations are compatible with the  $D$ -comodule structure on  $C$ . Thus there is a spectral sequence

$$E_1^{i,j} = H^{i,j}(\text{gr}_N D, \text{gr}_N C) \implies H^i(D, C),$$

where the structure of graded  $\mathrm{gr}_N D$ -comodule on  $\mathrm{gr}_N C$  coincides with the one induced by the morphism of graded coalgebras  $\mathrm{gr}_N g: \mathrm{gr}_N C \longrightarrow \mathrm{gr}_N D$ . Therefore, we have  $H^t(D, C) = 0$  provided that  $H^t(\mathrm{gr}_N D, \mathrm{gr}_N C) = 0$ . According to Theorem 3, there are natural isomorphisms  $\mathrm{gr}_N C \simeq A^?$  and  $\mathrm{gr}_N D \simeq B^?$ . Now the first statement follows from Theorem 7; it only remains to notice that for any comodule  $P$  over a conilpotent coalgebra  $C$  one has  $H^i(C, P) = 0$  for all  $i \geq t$  whenever  $H^t(C, P) = 0$ . The latter property is deduced from the fact that  $H^0(C, P) = 0$  implies  $P = 0$ .

In general, the coaugmentation filtration on the coalgebra  $C$  can differ from the coaugmentation filtration defined on the space  $C$  as a  $D$ -comodule. One can see that the two filtrations coincide if and only if the morphism  $C \longrightarrow D$  is injective, or, equivalently, the morphism  $A_1 \longrightarrow B_1$  is injective. By Theorem 4, in this case the  $B$ -module  $H^*(D, C)$  is Koszul if and only if the  $\mathrm{gr}_N D$ -comodule  $\mathrm{gr}_N C$  is Koszul. The last assertion now follows from Theorem 3, Corollary 5(a), and the version of Corollary 3(a) for morphisms of graded coalgebras.  $\square$

**Corollary 9.** *Let  $\psi: G' \longrightarrow G''$  be a homomorphism of pro- $l$ -groups and  $G$  be its kernel. Assume that the cohomology algebras  $A = H^*(G'', \mathbb{Z}/l)$  and  $B = H^*(G', \mathbb{Z}/l)$  are Koszul. In this case, if the  $A$ -module  $B$  is Koszul, then the homomorphism  $\psi$  is injective; if  $H_{i,j}(A, B) = 0$  for all  $i - j \neq 0, 1$ , then the group  $G$  is a free pro- $l$ -group.*

*Proof:* For any morphism of pro-finite groups  $\psi: G' \longrightarrow G''$  with the kernel  $G$  the natural morphism of the group coalgebras  $\mathbb{Z}/l(G'') \longrightarrow \mathbb{Z}/l(G')$  defines the structures of left and right comodule over the coalgebra  $\mathbb{Z}/l(G')$  on the vector space  $\mathbb{Z}/l(G'')$ . Just as in the proof of Corollary 6, one can see that the coalgebra  $\mathbb{Z}/l(G'')$  is a cofree comodule over  $\mathbb{Z}/l(G'/G)$ , the comodule  $\mathbb{Z}/l(G'/G)$  over the coalgebra  $\mathbb{Z}/l(G')$  is induced from the trivial  $\mathbb{Z}/l(G)$ -comodule, and it follows that  $H^*(\mathbb{Z}/l(G'), \mathbb{Z}/l(G'')) \simeq H^*(\mathbb{Z}/l(G'), \mathbb{Z}/l(G'/G)) \odot_{\mathbb{Z}/l(G'/G)} \mathbb{Z}/l(G'') \simeq H^*(G, \mathbb{Z}/l) \odot_{\mathbb{Z}/l(G'/G)} \mathbb{Z}/l(G'')$ . So one has  $H^i(G, \mathbb{Z}/l) = 0$  if and only if  $H^i(\mathbb{Z}/l(G'), \mathbb{Z}/l(G'')) = 0$ . It remains to apply Theorem 8; one does not have to distinguish  $H^{i,j}(A, B)$  from  $H^{i,j}(A^{\mathrm{op}}, B^{\mathrm{op}})$  because the pro-finite group cohomology algebras are (skew)-commutative.  $\square$

**Theorem 9.** *Let  $G'$  be a pro- $l$ -group,  $G \subset G'$  a normal subgroup, and  $G'' = G'/G$  the quotient group. Denote by  $A = H^*(G'', \mathbb{Z}/l)$  and  $B = H^*(G', \mathbb{Z}/l)$  the cohomology algebras of the quotient group  $G''$  and the group  $G'$ . Suppose that*

- (1) *the natural homomorphism  $f: A \longrightarrow B$  is an isomorphism in degree 1 and an epimorphism in degree 2;*
- (2) *the kernel  $J$  of the morphism  $f$  is isomorphic to the shift  $K(2)$  of a graded  $A$ -module  $K$  for which the morphism  $r_{A,K}: q_A K \longrightarrow K$  is an isomorphism in degree 1 and a monomorphism in degree 2;*
- (3) *the algebra  $A$  is Koszul and the  $A$ -module  $q_A K$  is Koszul.*

*Then the subgroup  $G$  is a free pro- $l$ -group.*

*Proof:* Consider the Serre–Hochschild spectral sequence

$$E_2^{p,q} = H^p(G'', H^q(G, \mathbb{Z}/l)) \implies H^{p+q}(G', \mathbb{Z}/l), \quad d_r^{p,q}: E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}.$$

The operator  $d_2: E_2^{p-2,1} \rightarrow E_2^{p,0}$  together with the projection of  $E_2^{p,0}$  into  $E_\infty^{p,0}$  define a 3-term complex of graded modules over  $E_2^{*,0} = H^*(G'', \mathbb{Z}/l)$

$$H^*(G'', H^1(G, \mathbb{Z}/l))(2) \longrightarrow H^*(G'', \mathbb{Z}/l) \longrightarrow H^*(G', \mathbb{Z}/l).$$

Let us denote by  $P$  the  $G''$ -module  $H^1(G, \mathbb{Z}/l)$ . Taking the kernel of the second arrow of this complex, we get a morphism of graded  $A$ -modules  $H^*(G'', P) \rightarrow K$ .

Since the morphism  $H^*(G'', \mathbb{Z}/l) \rightarrow H^*(G', \mathbb{Z}/l)$  is surjective in the degrees 1 and 2, one has  $E_3^{0,1} = E_3^{1,1} = 0$ . It follows easily that the map  $H^0(G'', P) \rightarrow K_0$  is an isomorphism and the map  $H^1(G'', P) \rightarrow K_1$  is injective. Therefore, the quadratic parts of the  $A$ -modules  $H^*(G'', P)$  and  $K$  coincide. Considering the composition

$$q_A K \simeq q_A H^*(G'', P) \longrightarrow H^*(G'', P) \longrightarrow K$$

and using the condition (2), we deduce that the map  $H^1(G'', P) \rightarrow K_1$  is an isomorphism and the morphism  $r_{A, H^*(G'', P)}: q_A H^*(G'', P) \rightarrow H^*(G'', P)$  is an isomorphism in degree 1 and a monomorphism in degree 2.

According to Theorem 4, it follows from the latter that the  $A$ -module  $H^*(G'', P)$  is quadratic, since the algebra  $A$  and the  $A$ -module  $q_A H^*(G'', P) \simeq q_A K$  are Koszul. The above composition now shows that the map  $H^2(G'', P) \rightarrow K_2$  is injective. Returning back to the spectral sequence, we conclude that the morphism  $d_2^{0,2}: E_2^{0,2} \rightarrow E_2^{2,1}$  is zero because the map  $H^2(G'', P) \rightarrow K_2$  is injective and that the morphism  $d_3^{0,2}: E_3^{0,2} \rightarrow E_3^{3,0}$  is zero because the map  $H^1(G'', P) \rightarrow K_1$  is surjective.

Because the morphism  $H^2(G'', \mathbb{Z}/l) \rightarrow H^2(G', \mathbb{Z}/l)$  is surjective, we have  $E_4^{0,2} = 0$  and it now follows that  $E_2^{0,2} = 0$ . This means that  $H^0(G'', H^2(G, \mathbb{Z}/l)) = 0$ . Since  $G''$  is a pro- $l$ -group, we have  $H^2(G, \mathbb{Z}/l) = 0$ . Finally, since the group  $G$  is a pro- $l$ -group, it is a free pro- $l$ -group according to a result from [18].  $\square$

**Lemma 3.** *Let  $F$  be a field containing a primitive  $l$ -root of unity if  $l$  is odd and containing a square root of  $-1$  if  $l = 2$ . Then the cohomology algebra of the Galois group  $\text{Gal}(F[\sqrt[l]{F}]/F)$  with constant coefficients  $\mathbb{Z}/l$  is naturally isomorphic to the exterior algebra  $\Lambda_l^*(F) = \bigwedge_{\mathbb{Z}/l}^*(F^*/F^{*l})$ .*

*Proof:* By Kummer's theory, it is clear that the first cohomology group of  $\text{Gal}(F[\sqrt[l]{F}]/F)$  is isomorphic to  $F^*/F^{*l}$ . The square root of  $-1$  condition guarantees that the cohomology algebra is skew-commutative. It suffices to show that the cohomology algebra is the exterior algebra over the first cohomology group.

Introduce the notation for fields  $K = F[\sqrt[l]{F}]$  and  $E = F[\sqrt[l]{1}]$ . By Kummer's theory, the Galois group  $\text{Gal}(K/E)$  is naturally a subgroup of the group of homomorphisms from  $F^*$  to the projective limit of the groups of  $l^N$ -roots of unity,  $\varprojlim \mu_{l^N}$ . The group  $\text{Hom}(F^*, \varprojlim \mu_{l^N})$  is an abelian pro- $l$ -group without torsion, hence so is the group  $\text{Gal}(K/E)$ . It follows that  $\text{Gal}(K/E)$  is a free abelian pro- $l$ -group and its cohomology ring is an exterior algebra.

If the field  $F$  contains all the  $l^N$ -roots of unity, then  $E = F$  and we are done. Otherwise, the group  $\text{Gal}(E/F)$  is isomorphic to  $\mathbb{Z}_l$  and it acts trivially in the cohomology algebra of  $\text{Gal}(K/E)$ . The Serre–Hochschild spectral sequence degenerates



into a series of exact sequences of the form

$$0 \rightarrow H^{n-1}(\text{Gal}(K/E), \mathbb{Z}/l) \longrightarrow H^n(\text{Gal}(K/F), \mathbb{Z}/l) \longrightarrow H^n(\text{Gal}(K/E), \mathbb{Z}/l) \rightarrow 0.$$

It follows easily that  $H^*(\text{Gal}(K/F), \mathbb{Z}/l)$  is an exterior algebra.  $\square$

*Proof of Theorem 2:* It follows from the condition (1) of Theorem 2 that the quadratic part  $qH^*(G_F, \mathbb{Z}/l)$  of the Galois cohomology algebra is isomorphic to the Milnor algebra  $K_*^M(F) \otimes \mathbb{Z}/l$ . By Theorem 6, it follows from the condition (2) that the latter algebra is Koszul. Therefore, Corollary 2 yields the isomorphisms

$$H^*(G_F^{(l)}, \mathbb{Z}/l) \simeq qH^*(G_F, \mathbb{Z}/l) \simeq K_*^M(F) \otimes \mathbb{Z}/l.$$

Now it remains to apply Theorem 9 or Corollary 9 to the pro- $l$ -group  $G' = G_F^{(l)}$  with the quotient group  $G'' = \text{Gal}(K/F)$  and the corresponding subgroup  $G = \text{Gal}(F^{(l)}/K)$  (where  $F^{(l)}$  is the maximal Galois pro- $l$ -extension of  $F$ ). It is obvious that  $G$  is isomorphic to the maximal quotient pro- $l$ -group of the absolute Galois group of  $K$  that we are interested in.  $\square$

## APPENDIX. SUPPORTING EVIDENCE FOR CONJECTURE 1

In this section we present a counterexample showing that the most straightforward generalization of Bogomolov's conjecture to arbitrary fields fails and prove several results providing evidence in support of our Conjecture 1.

**Counterexample.** *There exist fields  $F$  such that the commutator subgroup of the maximal quotient pro- $l$ -group of the absolute Galois group of  $F$  is not a free pro- $l$ -group. Moreover, there are fields such that the commutator subgroup of their Sylow pro- $l$ -subgroup of the absolute Galois group is not free.*

*Proof:* Let  $F$  be a field of characteristic not equal to  $l$  which contains a primitive root of unity of degree  $l$  but no primitive roots of unity of degree  $l^N$  for some  $N > 1$  (assumed to be the minimal such integer). Suppose that the maximal quotient pro- $l$ -group of  $G_F$  is not abelian. It is very easy to find a number field,  $l$ -adic field, or power series field satisfying the listed properties.

Consider the field  $F((z))$  of formal Laurent power series in the variable  $z$  with coefficients in  $F$ . Then the maximal quotient pro- $l$ -group of the absolute Galois group of the field of power series  $G_{F((z))}^{(l)}$  is isomorphic to the semidirect product of the group  $G_F^{(l)}$  with the group of  $l$ -adic integers  $\mathbb{Z}_l$ , where  $G_F^{(l)}$  acts via the cyclotomic character. The commutator subgroup of this semidirect product group is the semidirect product of the commutator subgroup of  $G_F^{(l)}$  with  $l^N \mathbb{Z}_l$ . This is clearly not a free pro- $l$ -group. The Sylow pro- $l$ -subgroup of the absolute Galois group of  $F((z))$  is isomorphic to the semidirect product of the Sylow subgroup of  $G_F$  with  $\mathbb{Z}_l$ . The commutator subgroup of this group is described as above; it is not a free pro- $l$ -group either.

In particular, for any field  $F$  containing a primitive  $l$ -root of unity but no primitive  $l^N$ -roots of unity for some  $N > 1$ , the field of doubly iterated formal Laurent power series  $F((w))((z))$  provides the desired counterexample.  $\square$

**Proposition A.1.** *Conjecture 1 holds for number fields and their functional analogues, i. e., algebraic extensions of  $\mathbb{Q}$  and  $\mathbb{F}_q(x)$ . Moreover, in these cases it suffices to add roots of unity only, i. e., take  $K = F[\sqrt[l^\infty]{1}]$ .*

*Proof:* The assertion is (essentially) a particular case of the result of [18, II.3.3]. It suffices to prove that  $H^2(G_M, \mathbb{Z}/l) = 0$  for any field  $M$  between  $K$  and  $\overline{F}$ . For any field  $L$  containing a primitive  $l$ -root of unity, the group  $H^2(G_L, \mathbb{Z}/l)$  is isomorphic to the subgroup of the Brauer group  $\text{Br}(L)$  annihilated by  $l$ . Obviously, the Brauer group functor preserves direct limits of fields. So it suffices to show that for any field  $E$  finite over  $\mathbb{Q}$  or  $\mathbb{F}_q(x)$  and any element  $\alpha \in \text{Br}(E)$  there exists a field  $L$  finite over  $E$  and contained in the composite  $EK$  such that the image of  $\alpha$  in  $\text{Br}(L)$  vanishes.

It remains to use the description of Brauer groups given by class field theory. The Brauer group of a global field is a subgroup of the direct sum of the Brauer groups of local completions. For any class  $\alpha \in \text{Br}(E)$  there is a finite number of completions  $E_v$  of the field  $E$  where the local class  $\alpha_v$  is nonzero. Adjoining  $l^N$ -roots of unity with large enough  $N$  to a nonarchimedean complete field  $E_v$ , one can obtain extensions of  $E_v$  of degrees divisible by arbitrary high powers of  $l$  (since  $E_v$  is finite over a field  $\mathbb{Q}_p$  or  $\mathbb{F}_q[[z]]$ , which has this property). And by local class field theory an  $l$ -torsion element of  $\text{Br}(E_v)$  dies in any finite extension of  $E_v$  of degree divisible by  $l$ . In the archimedean case, the local Brauer group is zero (for  $E_v = \mathbb{C}$ ) or  $\mathbb{Z}/2$  (for  $E_v = \mathbb{R}$ ), and it suffices to adjoin the square root of  $-1$  in order to kill it.  $\square$

**Proposition A.2.** *Let  $F$  be a Henselian discrete valuation field and  $f$  be its residue field. Then the field  $F$  satisfies Conjecture 1 whenever  $f$  does (for a given  $l$ ). In particular, if  $\text{char } f = l$ , then  $F$  satisfies Conjecture 1.*

*Proof:* As in the proof of Proposition A.1, we will argue in terms of the Brauer groups. The Brauer groups of Henselian discrete valuation fields with perfect residue fields were computed in [19, chapitre XII] and [13, §2].

Consider the two cases  $\text{char } f \neq l$  and  $\text{char } f = l$  separately. It is easy to verify following the arguments of [19] and [13] that for  $\text{char } f \neq l$  there is always a natural exact sequence

$$0 \longrightarrow \text{Br}(f)_{(l)} \longrightarrow \text{Br}(F)_{(l)} \longrightarrow \text{Hom}(G_f, v(F) \otimes \mathbb{Q}/\mathbb{Z})_{(l)} \longrightarrow 0,$$

where  $v(F)$  denotes the group of values of the discrete valuation,  $G_f$  is the absolute Galois group of  $f$ , and the subindex  $l$  in parentheses signifies the  $l$ -primary components of the corresponding abelian groups. Let  $E$  be a finite extension of the field  $F$  and  $e$  be its residue field. We would like to prove that any element of  $\text{Br}(E)_{(l)}$  can be killed by a radical extension of  $F$ . Let  $\alpha \in \text{Br}(E)_{(l)}$ ; for compactness reasons, the corresponding element of  $\text{Hom}(G_e, v(E) \otimes \mathbb{Q}/\mathbb{Z})$  is a homomorphism with finite image. We kill this class by adjoining a root of an appropriate degree from a uniformizing element of the valuation  $v$  in  $F$ . This extends our field  $E$  to a larger field  $D$ , still finite over  $F$ ; let  $d$  denote its residue field. Applying the above exact sequence to the valued field  $D$ , we get an element  $\alpha' \in \text{Br}(d)_{(l)}$ . By our assumption, this element can be killed by adjoining  $l$ -power roots of elements of  $f$ . To make

the original element  $\alpha \in \text{Br}(E)_{(l)}$  vanish, it now suffices to adjoin the corresponding roots of any preimages in  $F$  of those residue elements, together with the root of the uniformizing element in  $F$  that we adjoined already.

In the case  $\text{char } f = l$  the main step is to reduce the problem to the situation of a perfect residue field. Let us repeat transfinitely the following process of constructing purely inseparable extensions  $e$  of the field  $f$ . Choose any element of  $f$  which is not an  $l$ -power in  $f$  and adjoin to  $f$  all roots of this element of degrees  $l^N$ ; this is the first step. On each of the subsequent steps we choose an element of  $f$  which is still not an  $l$ -power in  $e$  and adjoin all its  $l^N$ -degree roots to  $e$ . When the transfinite induction terminates, we have a purely inseparable extension  $d$  of the field  $f$  which is easily found to be its perfect closure. Indeed, any  $x \in f$  is an  $l$ -power in  $d$  by construction; let us assume that it is an  $l^{n-1}$ -power and show that it is also an  $l^n$ -power. Let  $x = y^{l^{n-1}}$ ,  $y \in d$ . The element  $y$  can be expressed in terms of elements of  $f$  and roots of elements of  $f$  that we adjoined in the process. But both kinds of elements are  $l$ -powers in  $d$  by the construction.

Now we repeat this transfinite process on the level of valued fields. At each step we choose a compatible family of roots of all degrees  $l^N$ , one root of each degree, taken from an arbitrary element of  $F$  lifting  $x \in f$ . Notice that we avoid any extension of the group of values of the valuation; indeed, at each step the total degree of the extension coincides with the degree of extension of the residue fields. So the valuation remains discrete. The transfinite process results in the construction of a Henselian discrete valuation field  $D$  with a perfect residue field. According to [19] and [13], we have again an exact sequence

$$0 \longrightarrow \text{Br}(d) \longrightarrow \text{Br}(D) \longrightarrow \text{Hom}(G_d, v(D) \otimes \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

In addition, the  $l$ -primary component of the Brauer group of a perfect field of characteristic  $l$  vanishes. By construction, we have  $F \subset D \subset F[l^\infty\sqrt{F}]$ .

Finally, let  $M$  be a finite extension of the field  $D$  and  $m$  be its residue field. It remains to annihilate the class from  $\text{Hom}(G_m, v(M) \otimes \mathbb{Q}/\mathbb{Z})_{(l)}$ . As above, we do this by adjoining a root of high enough  $l$ -power degree from a uniformizing element of the valuation  $v$  of the field  $F$ .  $\square$

**Remark 1:** Here is another partial result in support of Conjecture 1, provided by a version of an argument which I learned from F. Bogomolov sometime in 1995–96. Let  $F$  be a field of characteristic not equal to  $l$ ; set  $K = F[l^\infty\sqrt{F}]$ . The conjecture can be rephrased by saying that for any finite extension  $E$  of the field  $F$  the image of the group  $K_2^M(E)/l$  in  $K_2^M(KE)/l$  vanishes. Let us show that this is true for any finite Galois extension  $E/F$  with the Galois group isomorphic to a direct sum of copies of  $\mathbb{Z}/2$ , or a direct sum of copies of  $\mathbb{Z}/3$ , or, more generally, for any abelian field extension with the Galois group annihilated by a number  $q$  such that the field  $F$  has no nontrivial finite extensions of degrees less or equal to  $q/2$ . In particular, the assertion applies to any extension  $E/F$  with the Galois group  $\text{Gal}(E/F) \simeq \mathbb{Z}/l \oplus \cdots \oplus \mathbb{Z}/l$  of a field  $F$  having no finite extensions of the degrees prime to  $l$ .

The map  $K_2^M(KE)/l \longrightarrow K_2^M(KE) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  being injective, it suffices to prove that the image of the group  $K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  in  $K_2^M(KE) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  vanishes. Now  $K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q}$  is a representation of the finite group  $\text{Gal}(E/F)$  over a field of characteristic zero; so it decomposes into a direct sum of irreducibles. Moreover, any element of  $K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q}$  invariant under the action of a subgroup of  $\text{Gal}(E/F)$  comes from an element of  $K_2^M(L) \otimes_{\mathbb{Z}} \mathbb{Q}$  for the corresponding subfield  $F \subset L \subset E$  (which can be obtained using the transfer operation). If the Galois group  $\text{Gal}(E/F)$  is abelian, its irreducible representations are actually representations of its cyclic quotient groups, so their elements come from cyclic extensions  $L/F$  of degrees dividing  $q$ .

By the Bass–Tate lemma, for any extension  $L/F$  of degree  $\leq q$  of a field  $F$  having no nontrivial finite extensions of degrees  $\leq q/2$ , the multiplication map  $K_1^M(L) \otimes_{\mathbb{Z}} K_1^M(F) \longrightarrow K_2^M(L)$  is surjective [15, section 4]. Hence the map  $K_1^M(L) \otimes_{\mathbb{Z}} K_1^M(F) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow K_2^M(L) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  is surjective, too. Given that the map  $K_1^M(F) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow K_1^M(K) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  is zero by definition, it follows that the map  $K_2^M(L) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow K_2^M(KL) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  induced by the embedding of fields  $L \longrightarrow KL$  also vanishes.

Summing up the two previous paragraphs, since the group  $K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q}$  is generated by the images of the groups  $K_2^M(L) \otimes_{\mathbb{Z}} \mathbb{Q}$  for intermediate fields  $F \subset L \subset E$  of degree  $\leq q$  over  $F$  and the map  $K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  is obviously surjective, we can conclude that the map  $K_2^M(E) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow K_2^M(KE) \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l$  is zero. The same argument also applies, e. g., to nonabelian extensions  $E/F$  of any field  $F$  with the Galois group  $\text{Gal}(E/F)$  isomorphic to the symmetric group  $\mathbb{S}_3$ , as its only irreducible representation of dimension  $> 1$  over  $\mathbb{Q}$  is generated by its three 1-dimensional subspaces of invariants of the transpositions in  $\mathbb{S}_3$ , which correspond to intermediate subfields  $L$  of degree 3 over  $F$ .

**Remark 2:** The following assertion describing the absolute Galois groups of fields as being “close to free profinite groups” in a sense quite different from that of Conjectures 1–2 is not difficult to prove. Let  $F$  be a finitely generated field and  $G = \text{Gal}(\overline{F}/F)$  be its absolute Galois group. Then the group  $G$  admits a finite decreasing filtration  $G = G^{-1} \supset G^0 \supset G^1 \supset \dots \supset G^N \supset G^{N+1} = \{1\}$  by closed subgroups normal in  $G$  such that the quotient group  $G/G^0$  is isomorphic to either  $\{1\}$  or  $\mathbb{Z}/2$ , while all the subsequent quotient groups  $G^n/G^{n+1}$ ,  $n \geq 0$  are closed subgroups of free pro-finite groups. Moreover, the group  $G^0/G^1$  can be only nontrivial when the field  $F$  has characteristic 0 and does not contain the square root of  $-1$ . Replacing finite decreasing filtrations of profinite groups with transfinitely decreasing ones, one extends the result to arbitrary (infinitely generated) fields  $F$ .

It suffices to consider the case of a purely transcendental field  $F = \mathbb{F}_p(x_1, \dots, x_N)$  or  $\mathbb{Q}(x_2, \dots, x_N)$ . In the finite characteristic case, let us filter the field  $F$  by its subfields  $F_n = \mathbb{F}_p(x_1, \dots, x_{n-1})$  and the field  $\overline{F}$  by its subfields  $L_n = \overline{F}_n F$ . The fields  $F = L_0 \subset L_1 \subset L_2 \subset \dots \subset L_N \subset L_{N+1} = \overline{F}$  are intermediate fields between  $F$  and  $\overline{F}$ , normal over  $F$ . Set  $G_n = \text{Gal}(\overline{F}/L_n) \subset G = \text{Gal}(\overline{F}/F)$  for  $n = 0, \dots, N+1$ . The Galois group  $G^0/G^1 = \text{Gal}(L_1/F) \simeq \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \widehat{\mathbb{Z}}$  is a free pro-finite group with one

generator, and the Galois groups  $G^n/G^{n+1} = \text{Gal}(L_{n+1}/L_n) \simeq \text{Gal}(\overline{F_n(x_n)}/\overline{F_n(x_n)})$  for  $n \geq 1$  are known to be free pro-finite groups [22, Corollary 11.34].

In the zero characteristic case, let us first filter the field  $\overline{\mathbb{Q}}$  by its subfields  $\mathbb{Q} = \overline{\mathbb{Q}}_{-1} \subset \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \mathbb{Q}_2 = \overline{\mathbb{Q}}$ . Set  $\mathbb{Q}_0 = \mathbb{Q}[\sqrt{-1}]$ . To define the field  $\mathbb{Q}_1$ , consider the maximal abelian extension  $\mathbb{Q}^{\text{ab}} = \mathbb{Q}[\sqrt[n]{1}]$ . The Galois group  $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$  is naturally isomorphic to the product over prime numbers  $\widehat{\mathbb{Z}}^* = \prod_l \mathbb{Z}_l^*$  of the multiplicative groups of  $l$ -adic integers. Its maximal torsion subgroup is  $\widehat{\mathbb{Z}}_{\text{tors}}^* \simeq \mathbb{Z}/2 \times \prod_l (\mathbb{Z}/l)^*$ . Consider the intermediate field  $\mathbb{Q}'_1$  between  $\mathbb{Q}$  and  $\mathbb{Q}^{\text{ab}}$  corresponding to the closed subgroup  $\widehat{\mathbb{Z}}_{\text{tors}}^* \subset \widehat{\mathbb{Z}}^*$ , and set  $\mathbb{Q}_1 = \mathbb{Q}_0 \mathbb{Q}'_1$ . Then one has  $\text{Gal}(\mathbb{Q}_0/\mathbb{Q}) \simeq \mathbb{Z}/2$  and  $\text{Gal}(\mathbb{Q}_1/\mathbb{Q}_0) \simeq \widehat{\mathbb{Z}}$ . The pro-finite group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_1)$ , having cohomological dimension 1 by a slight strengthening of the result of Proposition A.1 above (provable by the same method), is a closed subgroup of a free pro-finite group [8, section 22.4].

Finally, we filter the field  $F$  by its subfields  $F_n = \mathbb{Q}(x_2, \dots, x_{n-1})$ ,  $n = 2, \dots, N+1$ , and the field  $\overline{F}$  by its subfields  $L_0 = \mathbb{Q}_0 F$ ,  $L_1 = \mathbb{Q}_1 F$ , and  $L_n = \overline{F_n} F$  for  $n \geq 2$ . The fields  $F = L_{-1} \subset L_0 \subset L_1 \subset \dots \subset L_N \subset L_{N+1} = \overline{F}$  are intermediate fields between  $F$  and  $\overline{F}$ , normal over  $F$ . Set  $G_n = \text{Gal}(\overline{F}/L_n) \subset G = \text{Gal}(\overline{F}/F)$  for  $n = -1, \dots, N+1$ . Then the group  $G^{-1}/G^0 = \text{Gal}(L_0/F) \simeq \text{Gal}(\mathbb{Q}_0/\mathbb{Q})$  is isomorphic to  $\mathbb{Z}/2$ , the group  $G^0/G^1 = \text{Gal}(L_1/L_0) \simeq \text{Gal}(\mathbb{Q}_1/\mathbb{Q}_0)$  is a free pro-finite group with one generator, the group  $G^1/G^2 = \text{Gal}(L_2/L_1)$  is isomorphic to  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_1)$ , and the groups  $G^n/G^{n+1} = \text{Gal}(L_{n+1}/L_n) \simeq \text{Gal}(\overline{F_n(x_n)}/\overline{F_n(x_n)})$  for  $n \geq 2$  are free pro-finite groups.

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